

Appendix

ENGR 12, Spring 2026.

Appendix

Units

In the SI system,

- The units of resistance are Ohms,
- The units of capacitance are Farads, and
- The units of inductance are Henrys.

Electrical quantities in the base units of kilograms, seconds, meters, and Amperes are not always very useful. However, it is useful to note that:

- Multiplying Ohms by Farads gives seconds, and
- Multiplying Henrys by Farads gives seconds squared.

First-order systems

A first-order system with transfer function $\frac{1}{\tau s + 1}$ has time constant τ . In connection with time constants, the following table may be useful.

The free response of a first-order system $\tau \dot{x} + x = f(t) = 0$ initialized at $x = x_0$ at integer multiples of τ is given by the following table.

Time	Value	Decimal	Percent
$t = 0\tau$	$x = x_0 e^{-0}$	$x = 1.0x_0$	100%
$t = 1\tau$	$x = x_0 e^{-1}$	$x = 0.3678x_0$	36.7%
$t = 2\tau$	$x = x_0 e^{-2}$	$x = 0.1353x_0$	13.5%
$t = 3\tau$	$x = x_0 e^{-3}$	$x = 0.0497x_0$	4.9%
$t = 4\tau$	$x = x_0 e^{-4}$	$x = 0.0183x_0$	1.8%
$t = 5\tau$	$x = x_0 e^{-4}$	$x = 0.0067x_0$	0.6%

Second-order systems

The second-order differential equation $m\ddot{x} + b\dot{x} + kx = f(t)$ can be represented using the transfer function $\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ if the input is $f(t)$ and the output kx , where the undamped natural frequency is

$$\omega_n = \sqrt{k/m}$$

and the damping ratio is

$$\zeta = \frac{b}{2\sqrt{mk}}.$$

The solution to the differential equation

$$m\ddot{x} + b\dot{x} + kx = 0$$

has three possible forms depending on whether the roots of the characteristic polynomial are:

1. Real and distinct: $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$, where λ_1 and λ_2 are the two roots of the characteristic polynomial
2. Complex: $x(t) = e^{rt} (c_1 \cos \omega_d t + c_2 \sin \omega_d t)$ where r and ω_d are the real and imaginary parts of the roots of the characteristic polynomial
3. Real and repeated: $x(t) = (c_1 + c_2 t) e^{\lambda t}$ where λ is the repeated root.

For underdamped systems,

- The damped natural frequency is $\omega_d = \omega_n \sqrt{1 - \zeta^2}$
- The resonant frequency is $\omega_d = \omega_n \sqrt{1 - 2\zeta^2}$
- The time constant is

$$\frac{1}{\zeta\omega_n}$$

Fourier Series

A periodic function $x(t)$ with period P is equal to the convergent infinite series

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \left(\frac{2\pi n t}{P} \right) + b_n \sin \left(\frac{2\pi n t}{P} \right) \right)$$

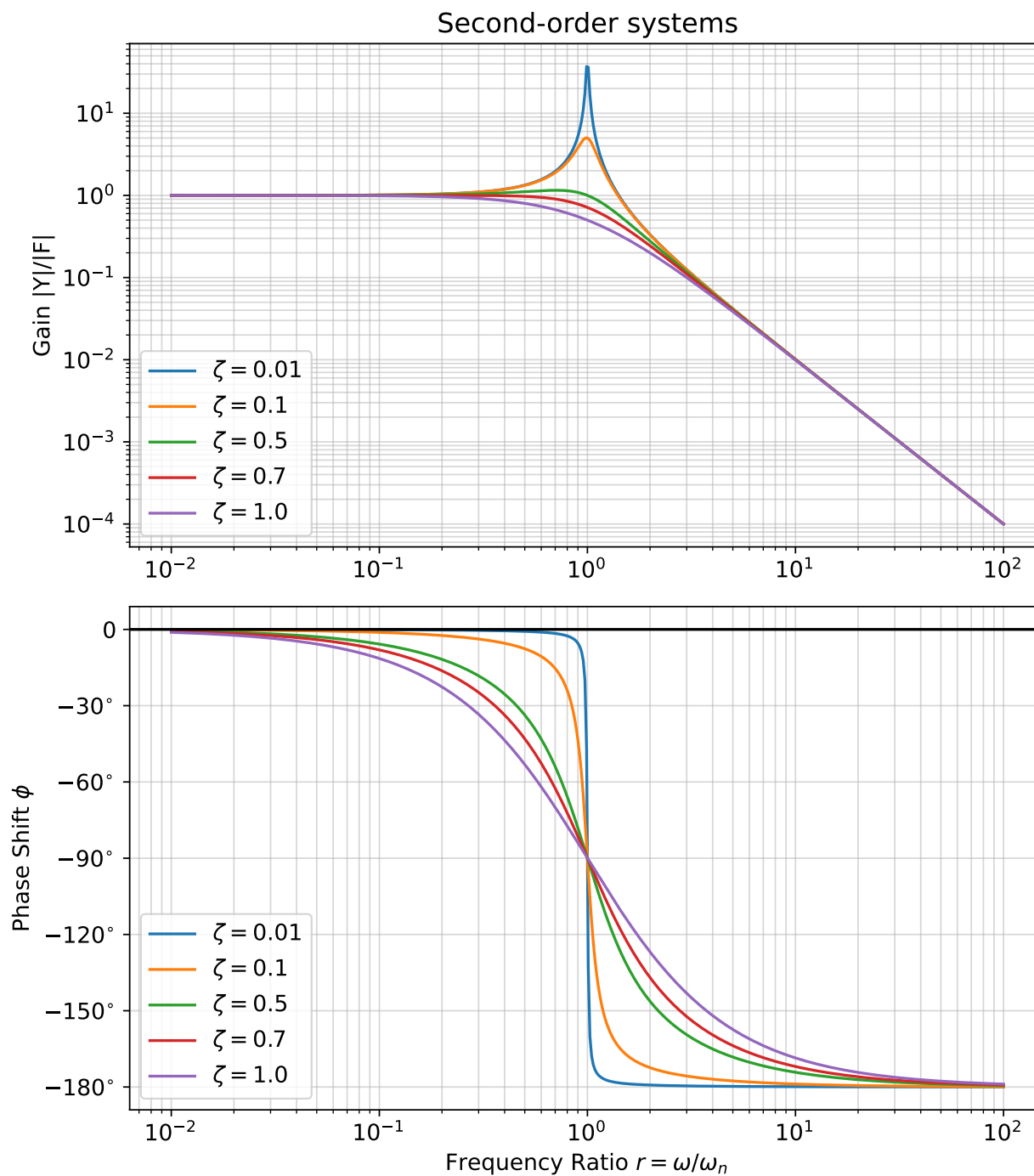
$$x(t) \approx a_0 + \sum_{n=1}^N \left(a_n \cos \left(\frac{2\pi n t}{P} \right) + b_n \sin \left(\frac{2\pi n t}{P} \right) \right)$$

for a_n , b_n given by the **Euler Formulas**:

$$a_0 = \frac{1}{P} \int_{-P/2}^{+P/2} x(t) dt$$

$$a_n = \frac{2}{P} \int_{-P/2}^{+P/2} x(t) \cos \left(\frac{2\pi n t}{P} \right) dt \quad n > 0$$

$$b_n = \frac{2}{P} \int_{-P/2}^{+P/2} x(t) \sin \left(\frac{2\pi n t}{P} \right) dt \quad n > 0$$

Bode Plots of Second-order systems

Partial Fraction Expansion

Consider the rational function

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0 s^0}{s^n + a_{n-1} s^{n-1} + \dots + a_0 s^0}$$

Distinct Poles

If $D(s)$ has n distinct roots, then we can use partial fractions to expand

$$F(s) = \frac{N(s)}{D(s)} = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + \dots + \frac{A_n}{s - s_n}$$

where

$$A_i = \lim_{s \rightarrow s_i} [(s - s_i)F(s)]$$

Repeated Poles

Let $F(s) = \frac{N(s)}{D(s)}$ where $D(s)$ has 2 repeated poles and $n - 2$ distinct poles. Then we can use partial fractions to expand

$$F(s) = \frac{N(s)}{D(s)} = \frac{A_{11}}{(s - s_1)^2} + \frac{A_{12}}{s - s_1} + \frac{A_3}{s - s_3} + \dots + \frac{A_n}{s - s_n}$$

where the coefficients $A_3, \dots, A_n$ can be found using a similar process as before:

$$A_i = \lim_{s \rightarrow s_i} [(s - s_i)F(s)]$$

and we have formulas for the numerators of the repeated poles:

$$A_{11} = \lim_{s \rightarrow s_1} [(s - s_1)^2 F(s)]$$

$$A_{12} = \lim_{s \rightarrow s_1} \left(\frac{d}{ds} [(s - s_1)^2 F(s)] \right)$$

Complex Poles

If $D(s)$ has two complex roots, then $F(s)$ can be expanded as

$$F(s) = \frac{K_1}{s - (r + i\omega)} + \frac{K_2}{s - (r - i\omega)}$$

where ω is the imaginary part of the complex roots and r the real part.

$$K_1 = \lim_{s \rightarrow r + i\omega} [(s - (r + i\omega))F(s)]$$

$$K_2 = \lim_{s \rightarrow r - i\omega} [(s - (r - i\omega))F(s)]$$

Laplace Transforms

The Laplace Transform of a function $x(t)$ is **defined** as the following function of s :

$$\lim_{T \rightarrow \infty} \int_0^T x(t) e^{-st} dt \quad (1)$$

This is a function of s , not a function of t . We give the expression in Equation 1 the name $X(s)$.

$$X(s) = \boxed{\lim_{T \rightarrow \infty} \int_0^T x(t) e^{-st} dt} = \int_0^\infty x(t) e^{-st} dt$$

$$x(t) \rightarrow \text{Laplace Transform} \rightarrow X(s)$$

$$x(t) \rightarrow \mathcal{L}[\cdot] \rightarrow X(s)$$

$$\boxed{\mathcal{L}[x(t)] = X(s)}$$

Table 3.3.2 Properties of the Laplace transform.

$x(t)$	$X(s) = \int_0^\infty f(t) e^{-st} dt$
1. $af(t) + bg(t)$	$aF(s) + bG(s)$
2. $\frac{dx}{dt}$	$sX(s) - x(0)$
3. $\frac{d^2x}{dt^2}$	$s^2X(s) - sx(0) - \dot{x}(0)$
4. $\frac{d^nx}{dt^n}$	$s^n X(s) - \sum_{k=1}^n s^{n-k} g_{k-1}$ $g_{k-1} = \left. \frac{d^{k-1}x}{dt^{k-1}} \right _{t=0}$
5. $\int_0^t x(t) dt$	$\frac{X(s)}{s} + \frac{g(0)}{s}$ $g(0) = \left. \int x(t) dt \right _{t=0}$
6. $x(t) = \begin{cases} 0 & t < D \\ g(t-D) & t \geq D \end{cases}$ $= u_s(t-D)g(t-D)$	$X(s) = e^{-sD}G(s)$
7. $e^{-at}x(t)$	$X(s+a)$

Laplace Tables

Table 3.3.1 Table of Laplace transform pairs.

$X(s)$	$x(t), t \geq 0$
1. 1	$\delta(t)$, unit impulse
2. $\frac{1}{s}$	$u_s(t)$, unit step
3. $\frac{c}{s}$	constant, c
4. $\frac{e^{-sD}}{s}$	$u_s(t - D)$, shifted unit step
5. $\frac{n!}{s^{n+1}}$	t^n
6. $\frac{1}{s + a}$	e^{-at}
7. $\frac{1}{(s + a)^n}$	$\frac{1}{(n - 1)!} t^{n-1} e^{-at}$
8. $\frac{b}{s^2 + b^2}$	$\sin bt$
9. $\frac{s}{s^2 + b^2}$	$\cos bt$
10. $\frac{b}{(s + a)^2 + b^2}$	$e^{-at} \sin bt$
11. $\frac{s + a}{(s + a)^2 + b^2}$	$e^{-at} \cos bt$
12. $\frac{a}{s(s + a)}$	$1 - e^{-at}$