Midterm Solutions

ENGR 12, Spring 2026.

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Solutions

1 Coupled vs Uncoupled Linear Equations

We are considering four systems, each of which have two state variables. The governing equation for each of the systems is as follows:

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Here, the matrix A is called the 'system matrix' and expresses compactly how the rate of change of the state variables depends on the state variables themselves.

Each of the four systems has a different system matrix A as shown in the table below. **Match the system to the correct description** by drawing lines from the matrix to its correct description. There is exactly one correct description for each matrix.

Solutions

No.	System Matrix	Description
1	$\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$	**3** y_1 and y_2 are two-way coupled, i.e., the rate of change of y_1 depends on y_2 and the rate of change of y_2 depends on y_1
2	$\begin{bmatrix} 1 & -1 \\ 0 & -3 \end{bmatrix}$	**2** y_1 and y_2 are one-way coupled, i.e., the rate of change of y_1 depends on y_2 but the rate of change of y_2 **does not** depend on y_1
3	$\begin{bmatrix} 2 & -4 \\ -2 & 7 \end{bmatrix}$	**4** y_1 and y_2 are one-way coupled, i.e., the rate of change of y_2 depends on y_1 but the rate of change of y_1 **does not** depend on y_2
4	$\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$	**1** y_1 and y_2 are uncoupled, i.e., the rate of change of y_1 does not depend on y_2 and the rate of change of y_2 does not depend on y_1

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2 Block Diagrams

Consider the block diagram below.

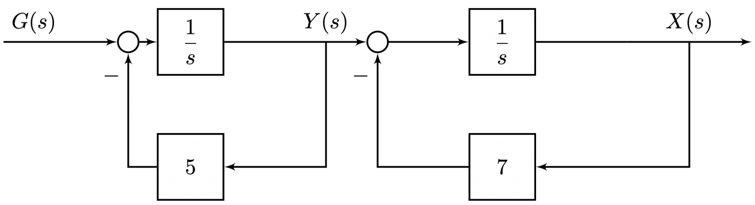


Figure 1

2.1 Simplifying

We would like to simplify this block diagram using the following block diagram:

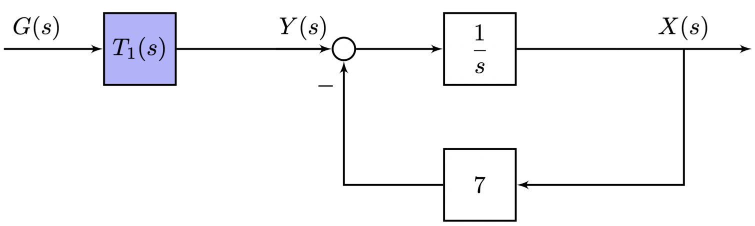


Figure 2

Determine the expression $T_1(s)$.

Solutions

$$\begin{aligned} sY(s) &= G(s) - 5Y(s) \\ sY + 5Y &= G \\ \frac{Y}{G} &= \frac{1}{s+5} \end{aligned}$$

so $\boxed{T_1(s) = \frac{1}{s+5}}$

2.2 Transfer Function

Determine an expression, in terms of s, for the transfer function between the input $G(s)$ and the output $X(s)$.

Solutions

$$\begin{aligned} sX(s) &= Y(s) - 7X(s) \\ sX + 7X &= Y \\ (s+7)X &= Y \\ X(s) &= \frac{1}{s+7}Y(s) \end{aligned}$$

and we already know from the previous part that

$$Y(s) = \frac{1}{s+5}G(s).$$

So we can combine these two to say that

$$\boxed{X(s) = \frac{1}{s+7}\frac{1}{s+5}G(s)}$$

3 Laplace Transforms

3.1 Converting from frequency domain to time domain.

Given

$$F(s) = \frac{1}{s(3s+2)},$$

find a mathematical expression for the function $f(t)$. Show all your work for full credit.

Solutions

We first notice that we can write the given expression as

$$F(s) = \frac{A}{s} + \frac{B}{3s+2}$$

where A and B are to be found. Let's calculate what they are:

$$\frac{A}{s} + \frac{B}{3s+2} = \frac{A(3s+2) + B \cdot s}{s(3s+2)}$$
$$\implies A(3s+2) + Bs = 1$$

and we can equate the coefficients of s^0 and s^1:

$$3As + Bs = 0, \quad 2A = 1$$
$$\implies A = \frac{1}{2}$$
$$3 \cdot \frac{1}{2} + B = 0 \implies B = -\frac{3}{2}$$

So we now have $F(s) = \dfrac{1/2}{s} + \dfrac{-3/2}{3s+2}$. But we notice that the table of Laplace transform pairs does not have terms of the form $3s+2$ in the denominator, but of the form $s+a$ in the denominator. So we re-write F as $F(s) = \dfrac{1/2}{s} + \dfrac{-3/2 \div 3}{s+2/3}$.
We are now in a position to read off the inverse Laplace Transform values from the table. We get

$$\mathcal{L}^{-1}\left[\frac{1/2}{s}\right] = \frac{1}{2}, \quad \mathcal{L}^{-1}\left[\frac{-1/2}{s+2/3}\right] = -\frac{1}{2}e^{-2t/3}$$

So

$$f(t) = \frac{1}{2} - \frac{1}{2}e^{-2t/3}$$

where it is assumed that both parts of this function 'start from' $t = 0$, hence we omit the $u_s(t)$ that should multiply **both** of these terms.

3.2 Converting from time domain to frequency domain

Find the transfer function for a third-order system given by

$$\dddot{x} = 2\ddot{x} - 4\dot{x} + 3x + f(t)$$

Solutions

We note that transfer functions are indifferent to the initial conditions. We can therefore take the Laplace Transform of the given equation, and get

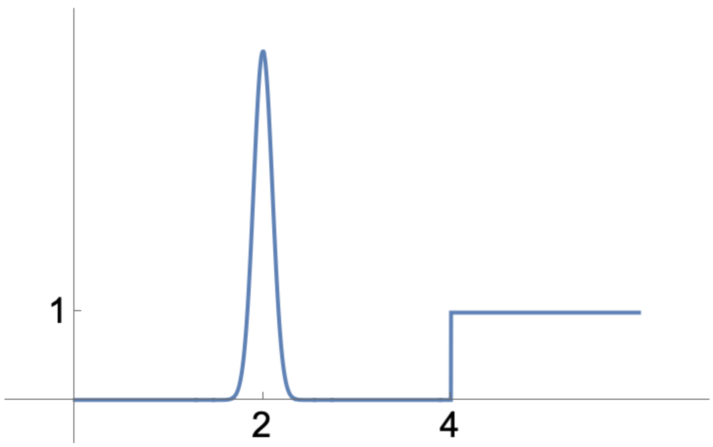
$$\begin{aligned}
\mathcal{L}^{-1}\left[\dddot{x}\right] &= \mathcal{L}[2\ddot{x}] - \mathcal{L}[4\dot{x}] + \mathcal{L}[3x] + \mathcal{L}\left[f(t)\right] \\
s^3 X(s) &= 2s^2 X(s) - 4sX(s) + 3X(s) + F(s) \\
\left(s^3 - 2s^2 + 4s - 3\right) X &= F \\
\frac{X}{F} &= \frac{1}{s^3 - 2s^2 + 4s - 3}
\end{aligned}$$

4 Impulse and Step Functions

4.1 Sketch

A certain system $a\dot{x} + bx = f(t)$ is subjected to a unit impulse input at time $t = 2$ and a unit step input at time $t = 4$. Make a qualitatively correct sketch of $f(t)$.

Solutions



4.2 Relation between impulse and step

Which of the following is true? There may be more than one answer.

Solutions

- ☒ The unit impulse function $\delta(t)$ is the derivative of the unit step function $u_s(t)$.
- ☐ The unit step function $u_s(t)$ is the derivative of the unit impulse function $\delta(t)$.
- ☐ Integrating the shifted unit step function $u_s(t-2)$ from $t = 2$ to $t = +\infty$ gives us the shifted unit impulse function $\delta(t-2)$.
- ☒ Integrating the shifted unit impulse function $\delta(t-2)$ from $t = -\infty$ to $t = +\infty$ gives us the shifted unit step function $u_s(t-2)$.

Note

Options 3 and 4 were not worded in the most mathematically rigorous way possible. Therefore, they have not been graded and everyone has received two points on this part automatically, regardless of which options were ticked between options 3 and 4.

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5 Inputs

Which of the following is a correct mathematical expression for the function of time shown below? There is only one correct answer.

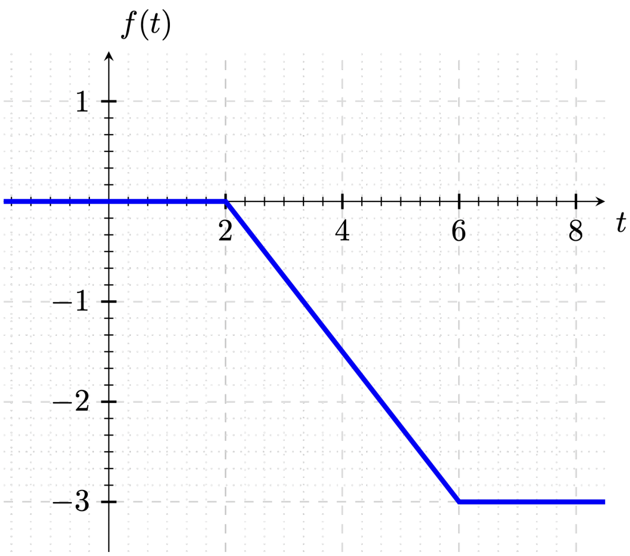

Figure 3

Solutions

- ☐ $f(t) = u_s(t-2) \times \left(-\frac{3}{4}(t-2)\right)$
- ☒ $f(t) = u_s(t-2) \times \left(-\frac{3}{4}(t-2)\right) + u_s(t-6) \times \left(\frac{3}{4}(t-6)\right)$
- ☐ $f(t) = u_s(t-2) \times \left(-\frac{4}{3}(t-2)\right) + u_s(t-6) \times \left(\frac{4}{3}(t-6)\right)$
- ☐ $f(t) = u_s(t-2) \times \left(\frac{3}{4}(t-2)\right) + u_s(t-6) \times \left(-\frac{3}{4}(t-6)\right)$
- ☐ $f(t) = u_s(t-2) \times \left(-\frac{3t}{4}\right) + u_s(t-6) \times \left(\frac{3t}{4}\right)$

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6 'Pulse response'

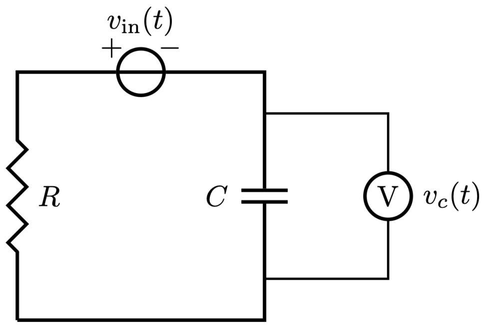


Figure 4: A circuit with governing equation $RC\dot{v}_c + v_c = v_{in}(t)$

An RC Circuit, shown in Figure 4, has a variable input voltage v_{in} given by a 'pulse' function

$$v_{in}(t) = \begin{cases} 0 & t < 2 \\ 2 & 2 \le t < 6 \\ 0 & t \ge 6. \end{cases}$$

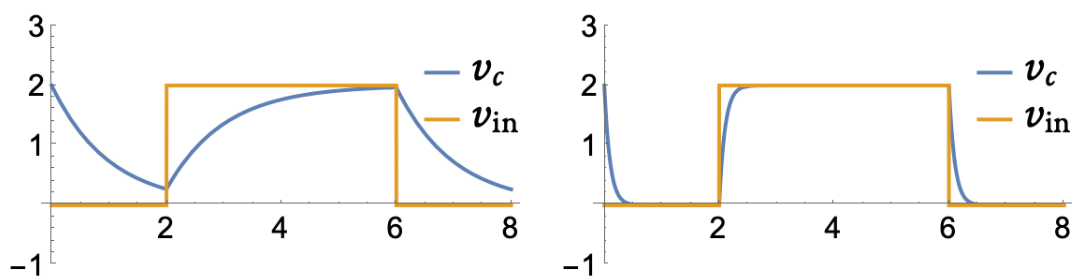
The voltage across the capacitor is initially 2.0 Volts.

Sketch a graph of the voltage across the capacitor, v_c, against time for the following cases:

Case	Resistance	Capacitance
1	1.25MΩ	0.8μF
2	1.25MΩ	0.08μF

using the graphs below.

Solutions



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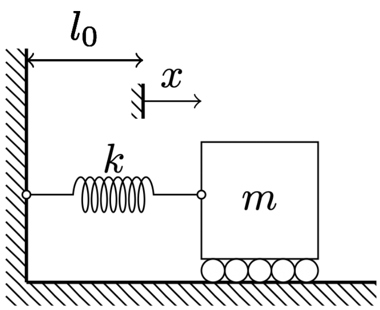
7 Energy

A spring-mass system is shown below.



The spring has a rest length of l_0, and the position coordinate x is defined such that, when $x = 0$, the spring exerts no force in either direction.

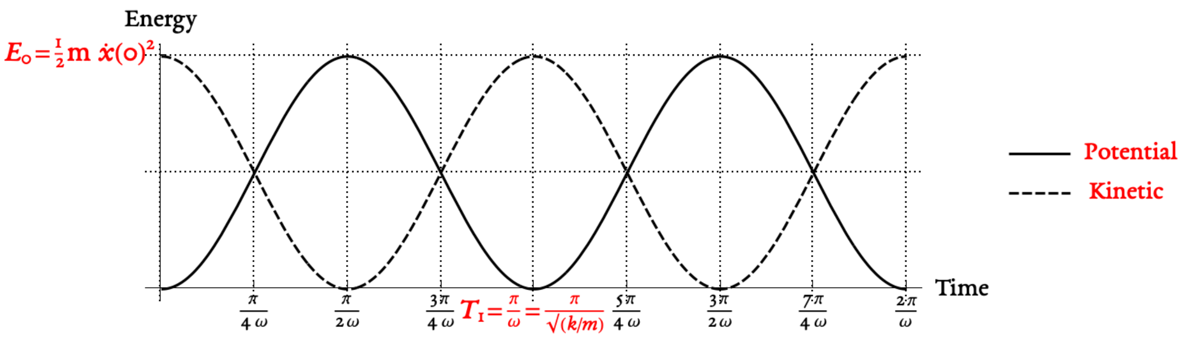
This system has the following initial conditions and parameters:

Quantity	Value
Initial Position	0 meters
Initial Velocity	+2.0 m/s
Spring Constant	5 N/m
Mass	0.2 kg

A graph of the energy in the system as a function of time is shown below.

1. Label the two lines on the graph with 'kinetic energy' and 'potential energy'.
2. Determine numerical values for E_0 and T_1.

Solution



The initial kinetic energy is $\frac{1}{2} \times 0.2 \times 2^2 = 0.4$ Joules.

The value of $T_1 = \pi/\omega = \pi \div \sqrt{\frac{k}{m}} = \pi \div \sqrt{\frac{5}{1/5}} = \frac{\pi}{5}$.

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8 Second-order systems with damping

A spring-mass damper system has the following differential equation:

$$m\ddot{x} + 2\dot{x} + 4x = f(t)$$

8.1 Characteristic polynomial

Write down the characteristic polynomial for this system in terms of m and s.

Solution

$$ms^2 + 2s + 4$$

8.2 Selecting the value of m

1. Write down a value of m for which the characteristic polynomial of the system will have two distinct negative roots.

Solution

$m = 1/5$

2. Write down a value of m for which the characteristic polynomial of the system will have two complex roots with negative real part.

Solution

$m = 1/3$

3. Suppose the system is initialized with no input f. For what value of m will the free response of this system die down to zero the quickest?

Solution

$m = 1/4$

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9 Appendix

9.1 First-order systems

The time constant in a first-order system

$$a\dot{x} + bx = f(t)$$

is given by the ratio a/b.

9.2 Second-order systems

The second-order differential equation

$$m\ddot{x} + b\dot{x} + kx = 0$$

can be represented by the equation

$$\ddot{x} + 2\zeta\omega_n x + \omega_n^2 x = 0$$

where the undamped natural frequency is

$$\omega_n = \sqrt{k/m}$$

and the damping ratio is

$$\zeta = \frac{b}{2\sqrt{mk}}.$$

9.3 Laplace Transfroms

The Laplace Transform of a function $x(t)$ is **defined** as the following function of s:

$$\lim_{T\to\infty} \int_0^T x(t)e^{-st}dt \tag{1}$$

This is a function of s, not a function of t. We give the expression in Equation 1 the name $X(s)$.

$$X(s) = \boxed{\lim_{T\to\infty} \int_0^T x(t)e^{-st}dt} = \int_0^\infty x(t)e^{-st}dt$$

$$x(t) \to \text{Laplace Transform} \to X(s)$$

$$x(t) \to \mathcal{L}[\cdot] \to X(s)$$

$$\boxed{\mathcal{L}[x(t)] = X(s)}$$

9.4 Laplace Tables

Table 3.3.1 Table of Laplace transform pairs.

	$X(s)$	$x(t), t \geq 0$
1.	1	$\delta(t)$, unit impulse
2.	$\frac{1}{s}$	$u_s(t)$, unit step
3.	$\frac{c}{s}$	constant, c
4.	$\frac{e^{-sD}}{s}$	$u_s(t-D)$, shifted unit step
5.	$\frac{n!}{s^{n+1}}$	t^n
6.	$\frac{1}{s+a}$	e^{-at}
7.	$\frac{1}{(s+a)^n}$	$\frac{1}{(n-1)!}t^{n-1}e^{-at}$
8.	$\frac{b}{s^2+b^2}$	$\sin bt$
9.	$\frac{s}{s^2+b^2}$	$\cos bt$
10.	$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
11.	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
12.	$\frac{a}{s(s+a)}$	$1-e^{-at}$

Table 3.3.2 Properties of the Laplace transform.

$x(t)$	$X(s) = \int_0^\infty f(t)e^{-st}\,dt$
1. $af(t) + bg(t)$	$aF(s) + bG(s)$
2. $\dfrac{dx}{dt}$	$sX(s) - x(0)$
3. $\dfrac{d^2x}{dt^2}$	$s^2X(s) - sx(0) - \dot{x}(0)$
4. $\dfrac{d^nx}{dt^n}$	$s^nX(s) - \sum_{k=1}^{n} s^{n-k}g_{k-1}$ $g_{k-1} = \left.\dfrac{d^{k-1}x}{dt^{k-1}}\right\rvert_{t=0}$
5. $\int_0^t x(t)\,dt$	$\dfrac{X(s)}{s} + \dfrac{g(0)}{s}$ $g(0) = \left.\int x(t)\,dt\right\rvert_{t=0}$
6. $x(t) = \begin{cases} 0 & t < D \\ g(t-D) & t \geq D \end{cases}$ $= u_s(t-D)g(t-D)$	$X(s) = e^{-sD}G(s)$
7. $e^{-at}x(t)$	$X(s+a)$