Problem Set 1 Solutions

ENGR 12, Spring 2026.

Due Date	Thu, Jan 29, 2026
Turn in link	Gradescope
Latest version	Course website

Note: 20% points on this assignment are for presentation and formatting of work.

1 Linearize a nonlinear 'system'

Consider the nonlinear function

$$f(x) = 5 + x(x-4)(x-2) \tag{1}$$

Derive the expressions for a 'linear model' of Equation 1 near...

1. $x = 1$
2. $x = 2$
3. $x = 4$

> **Linear Model**
>
> As used in this assignment, the term 'linear model' refers to a function $f^*(x)$ such that $f(x)$ is approximately equal to $f^*(x)$ in some part of the domain. $f^*(x)$ should be linear in x.

Write out your answers <u>including your derivation</u> by hand.

> **Typed vs. Handwritten**
>
> Neatly handwritten answers are expected here. You may choose to submit typed answers, but only if you use proper typesetting software like LaTeX.

Plot the functions that you find in the vicinity of each point, i.e., plot your answer to 1 in a small neighborhood of $x = 1$, your answer to 2 in a small neighborhood of $x = 2$, and your answer to 3 in a small neighborhood of $x = 4$. Overlay all three plots on a global plot of Equation 1 over a domain large enough to show the nonlinear behavior.

Getting comfortable with plotting

You may use any plotting software to do this; I recommend either `matplotlib.pyplot` in Python or the standard plotting commands in MATLAB. It is okay to use online resources to help you plot things, but your work must be your own.
If you need help getting started, check out the Resources page on the class website.

Solution

Derivation of linearized systems

The slope of the function $f(x)$ at any value of x is equal to its derivative evaluated at that value of x. With this in mind, note that the derivative of $f(x)$ is $f'(x) = 3x^2 - 12x + 8$.

1. At $x = 1$, $f'(x) = -1$
2. At $x = 2$, $f'(x) = -4$
3. At $x = 4$, $f'(x) = 8$

We now have the slopes of each of the three functions we need. But we also need their y-intercepts. To find those, note that you already have one point on that line.

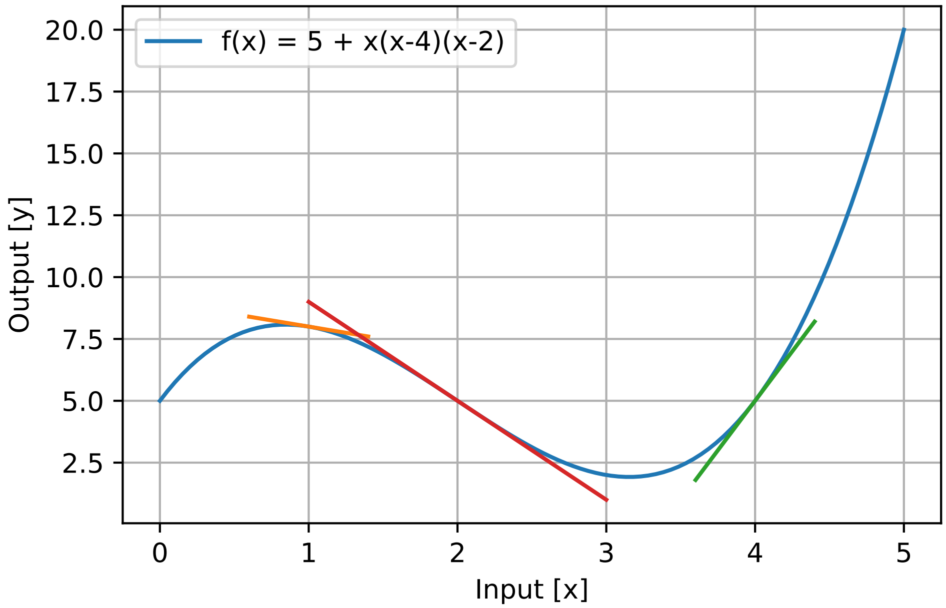
1. The point $(1, f(1)) = (1, 8)$ lies on the required line. Thus, we can write the equation
$$8 = (-1) \cdot (1) + c \implies c = 9$$
2. The point $(2, f(2)) = (2, 5)$ lies on the required line. Thus, we can write the equation
$$5 = (-4) \cdot (2) + c \implies c = 13$$
3. The point $(4, f(4)) = (4, 5)$ lies on the required line. Thus, we can write the equation
$$5 = (8) \cdot (4) + c \implies c = -27$$

Thus, our three functions are

1. $$y = -x + 9$$
2. $$y = -4x + 13$$
3. $$y = 8x - 27$$

Plotting it

```python
import numpy as np
import matplotlib.pyplot as plt

def f2(x): # a non-linear function
  return 5 + (x - 4)*(x)*(x - 2)

def f2prime(x): # the derivative of f2
  return 8 - 12*x + 3*x**2

# Create an 'x axis' with 100 points from 0 to 5
x_vals = np.linspace(0,5,100)

# Plot f2
plt.plot(x_vals,f2(x_vals),label="f(x) = 5 + x(x-4)(x-2)")

# Define the three points of interest
x1 = 1.0
x2 = 4.0
x3 = 2.0

# Create 'neighborhoods' around them.
x1vals = np.linspace(x1-0.4,x1+0.4,100)
x2vals = np.linspace(x2-0.4,x2+0.4,100)
x3vals = np.linspace(x3-1.0,x3+1.0,100)

# Define functions in their vicinity
f2x1 = lambda x: f2(x1)+f2prime(x1)*(x-x1)
f2x2 = lambda x: f2(x2)+f2prime(x2)*(x-x2)
f2x3 = lambda x: f2(x3)+f2prime(x3)*(x-x3)

# Plot
plt.plot(x1vals,f2x1(x1vals))
plt.plot(x2vals,f2x2(x2vals))
plt.plot(x3vals,f2x3(x3vals))

plt.xlabel("Input [x]")
plt.ylabel("Output [y]")
plt.legend()
plt.grid()
plt.show()
```


f(x) = 5 + x(x-4)(x-2)
Output [y]
Input [x]
20.0
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0
1
2
3
4
5

2 Complex Numbers

2.1 Cartesian and polar forms

The Cartesian form of a complex number is

$$z = x + iy, \tag{2}$$

and its polar form is

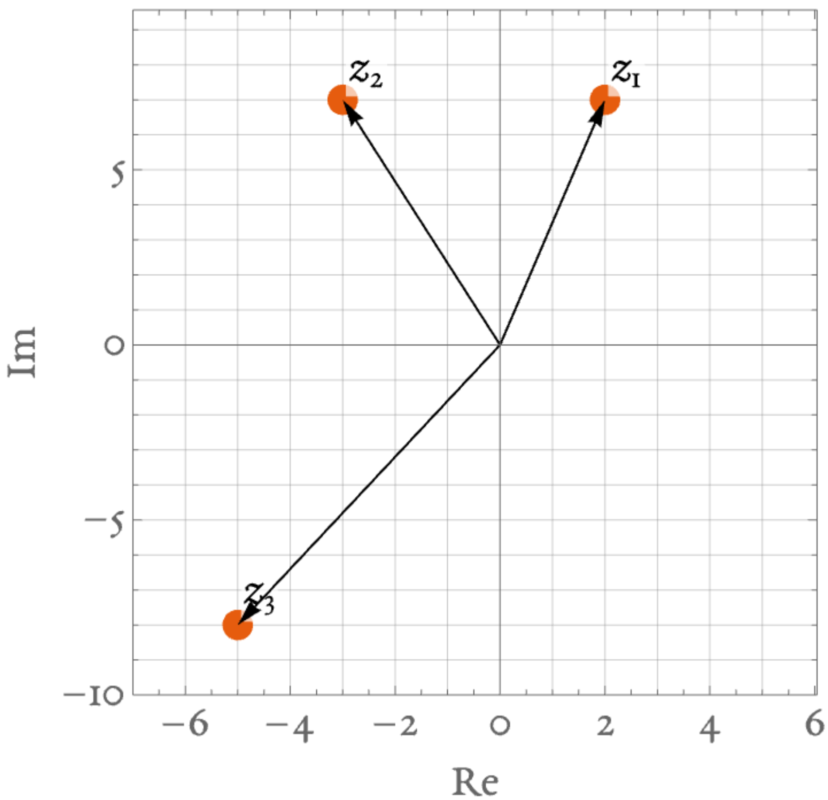
$$z = re^{i\theta} \tag{3}$$

1. Derive the polar form of the following complex numbers given in Cartesian form, i.e., calculate r and θ and then write it in a form like Equation 3

- $z_1 = 2 + 7i$
- $z_2 = -3 + 7i$
- $z_3 = -5 - 8i$

Solution

To calculate, recall the complex plane diagram from lecture 2 and note that $r = \sqrt{x^2 + y^2}$ and θ can be found from trigonometry.
Let's first plot all three on the complex plane.



We find that z_1 is in the first quadrant, z_2 in the second quadrant, and z_3 in the third quadrant. We can use this to put correct signs on the angle θ that we will find for each number.

1. Magnitude is $r = \sqrt{2^2 + 7^2} = \sqrt{53} \approx 7.28$. Argument is $\arctan 7/2 \approx 1.29$, so we have $z_1 \approx 7.28e^{1.29i}$.

2. Magnitude is $r = \sqrt{(-3)^2 + 7^2} = \sqrt{58} \approx 7.61$. Argument is $\pi - \arctan 7/2$ which is approximately 1.97, so we have $z_2 \approx 7.61e^{1.97i}$. (Why do we subtract from π? You should be able to figure this out)

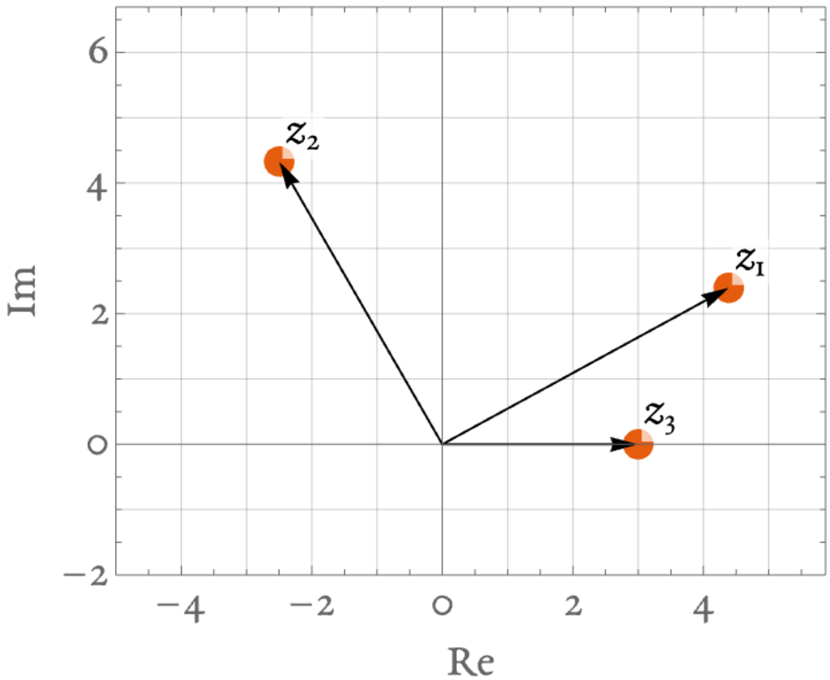
3. Magnitude is $r = \sqrt{(-5)^2} + (-8)^2 = \sqrt{89} \approx 9.43$. Argument is $\pi + \arctan 8/5 \approx 4.15$, so we have $z_3 \approx 9.43e^{4.15i}$. Note that you could subtract 2π from the argument and it would still be the same answer.

2. Derive the Cartesian form of the following complex numbers given in polar form, i.e., write it in the form of Equation 2.

- $z_4 = 5e^{0.5i}$
- $z_5 = 5ie^{i\pi/6}$
- $z_6 = 3e^0$

Solution

First, let's plot these on the complex plane.



We find that z_1 is in the first quadrant, z_2 in the second quadrant, and z_3 in the third quadrant. We can use this to put correct signs on the angle θ that we will find for each number.

4. The number z_4 can be written $r\cos\theta + ir\sin\theta$. Thus, we have $5\cos 0.5 + 5i\sin 0.5 \approx 4.38 + 2.39i$

5. The number z_5 should be placed into Cartesian form using a procedure such as follows.

$$\begin{aligned} z_5 &= i \times (5\cos\pi/6 + 5i\sin\pi/6) \\ &= i \times \left(\frac{5\sqrt{3}}{2} + i\frac{5}{2}\right) \\ &= -\frac{5}{2} + i\frac{5\sqrt{3}}{2} \end{aligned}$$

6. This is simply $3 + 0i$

2.2 Complex Arithmetic

Complex numbers can be added, subtracted, multiplied and divided.

Calculate the following complex numbers and give them in Cartesian or polar form.

1.

$$(-2)e^{0.3i} + 3e^{5i}$$

Solution

$$-1.05 - 3.46i$$

2.
$$2 + 5i - e^{-i\pi/4}$$

Solution

$$1.29 + 5.70i$$

3.
$$(5 + i) \times (3 - i)$$

Solution

$$16 - 2i$$

4.
$$2e^{3i} \times (-9)e^{7i}$$

Solution

$$15.10 + 9.79i$$

or

$$-18e^{10i}$$

5.
$$(6 + 7i) \div (7 - 6i)$$

Solution

$$0 + i$$

6.
$$-4e^{7i} \div 3e^{5i}$$

Solution

$$-\frac{4}{3}e^{2i}$$

or

$$0.55 - 1.21i$$

2.3 Plotting Complex Numbers

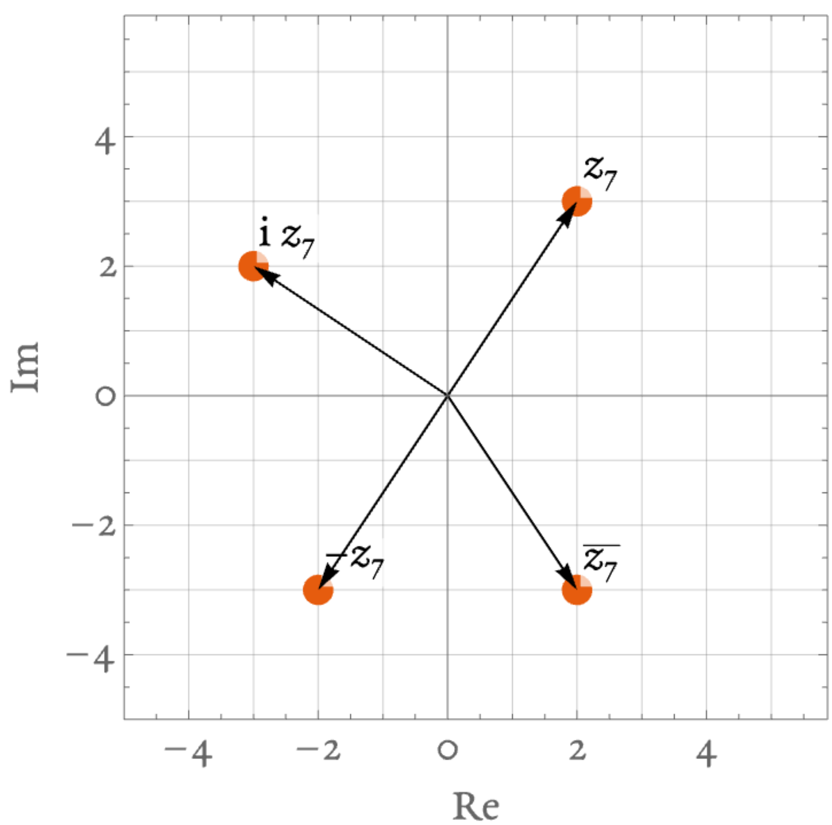
Start with the complex number $z_7 = 2 + 3i$. On the same set of coordinate axes representing the complex plane, plot

The complex conjugate

Remember that $\bar{z}$ is the complex conjugate of z, i.e., if $z = x + iy$ then $\bar{z} = x - iy$.

1. A point corresponding to z_7.
2. An arrow from the origin to z_7.
3. A point corresponding to iz_7.
4. An arrow from the origin to iz_7.
5. A point corresponding to $-z_7$.
6. An arrow from the origin to $-z_7$.
7. A point corresponding to $\bar{z}_7$.
8. An arrow from the origin to $\bar{z}_7$

Solution



3 Differential Equations

3.1 Placing equations in the standard form

The standard form for writing down a first-order differential equation is

$$\dot{x} = f(x, t). \tag{4}$$

Place the following equations in the form of Equation 4. In doing so, explicitly identify and write down the function $f(x, t)$.

> **Tip**
>
> Remember that
> $$f(x, t)$$
> could be a constant, just a function of t, just a function of x, or a function of *both*.

a.
$$\dot{x} + 2x = 1/x$$

> **Solution**
>
> $$\dot{x} = f(x, t) = \frac{1}{x} - 2x$$

b.
$$\dot{x}^2 + 2x = t$$

> **Solution**
>
> $$\dot{x} = f(x, t) = \sqrt{t - 2x}$$

c.
$$\dot{x}x + 2 = \cos t$$

> **Solution**
>
> $$\dot{x} = f(x, t) = \frac{(\cos t - 2)}{x}$$

d.
$$\dot{x} = e^x - e^t$$

> **Solution**
>
> $$\dot{x} = f(x, t) = e^x - e^t$$

e.

$$\dot{x} + \cos tx = 5$$

Solution

$$\dot{x} = f(x,t) = 5 - \cos tx$$

f.

$$\dot{x} = \frac{1}{x}$$

Solution

$$\dot{x} = f(x,t) = 1/x$$

3.2 Coupled and uncoupled differential equations

Which of the following sets of equations are coupled? State whether the coupling is in both ways, or only one way, and if the latter, then specify which way it is coupled. You do not need to show your work for this.

$$\begin{aligned} m_1 \frac{dv_1}{dt} + b_1 v_1 &= 0 \\ m_2 \frac{dv_2}{dt} + b_2 v_2 &= 0 \end{aligned} \tag{5}$$

Solution

The above equations are uncoupled.

$$\begin{aligned} m_1 \frac{dv_1}{dt} + b_1 v_1 &= 0 \\ m_2 \frac{dv_2}{dt} + b_2 (v_2 - v_1) &= 0 \end{aligned} \tag{6}$$

Solution

The above equations are coupled one way. v_2 depends on v_1 but v_1 does not depend on v_2.

$$\begin{aligned} m_1 \frac{dv_1}{dt} + b_1 (v_1 - v_2) &= 0 \\ m_2 \frac{dv_2}{dt} + b_2 v_2 &= 0 \end{aligned} \tag{7}$$

Solution

The above equations are coupled one way. v_1 depends on v_2 but v_2 does not depend on v_1.

$$\begin{aligned} m_1 \frac{dv_1}{dt} + b_1(v_1 - v_2) = 0 \\ m_2 \frac{dv_2}{dt} + b_2(v_2 - v_1) = 0 \end{aligned} \tag{8}$$

Solution

The above equations are coupled both ways. Both v_1 and v_2 depend on each other.

3.3 Linear and nonlinear differential equations

Which of the following differential equations are linear and which are nonlinear?

What linearity of a differential equation means

A differential equation is linear if the function f appearing on its right-hand side is linear in x and all its derivatives

a.
$$2\dot{x} + x = \cos t$$

Linear

b.
$$(\dot{x} + x)\cos x = t$$

Nonlinear

c.
$$(\dot{x} + x)\cos \pi/6 = 2t$$

Linear

d.
$$\dot{x} = \frac{1}{x}$$

Nonlinear

e.
$$\ddot{x} + 2\dot{x}^2 - 3x = 2t$$

Nonlinear

f.
$$\ddot{x} + 2\dot{x} - 3x = 2t$$

Linear

4 A coupled first-order linear system

Solving differential equations

is **not** necessary for this problem. We haven't gotten to that yet! The plots shown below were made by a computer program. Very soon, you'll learn how to make them yourself, but for now, don't worry about how the solution was arrived at.

In class, we discussed the system described by the following diagram.

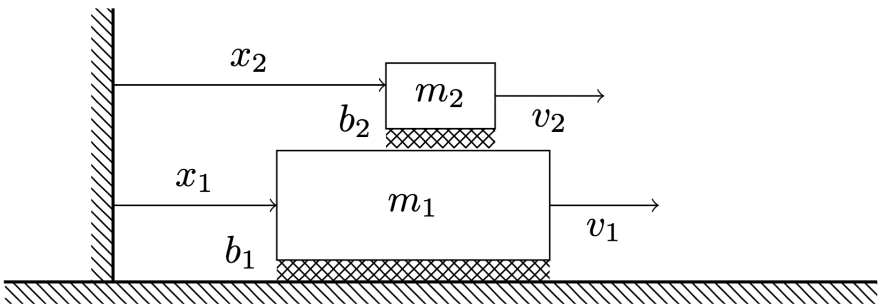


Figure 1: A box on a box with friction. Cross-hatch means a friction-full surface.

How long are the boxes?

We will assume that the upper box is so much smaller than the lower box that it will never fall off. The lower box is assumed to be on the ground and the ground goes on for ever.

The governing equations for this system, if it's not subjected to any external forces, are

$$m_1\dot{v}_1 + b_1 v_1 = 0$$
$$m_2\dot{v}_2 + b_2(v_2 - v_1) = 0$$

If you start this system off such that the bottom box has a starting positive nonzero speed, while the top box starts off with zero speed, it can be shown that the speeds of the two boxes v_1 and v_2 (in the directions shown in the diagram) will evolve over time as follows.

Note that box 1 is ten times heavier than box 2, and the two friction coefficients b_1 and b_2 are equal to each other.

4.1 Tasks

1. According to your intuition, what happens to the boxes' positions as measured from the fixed wall? Will x_1 and x_2, measured relative to the wall, both increase? Or will one of them increase and the other decrease? Describe what happens to the boxes' positions in a few sentences.

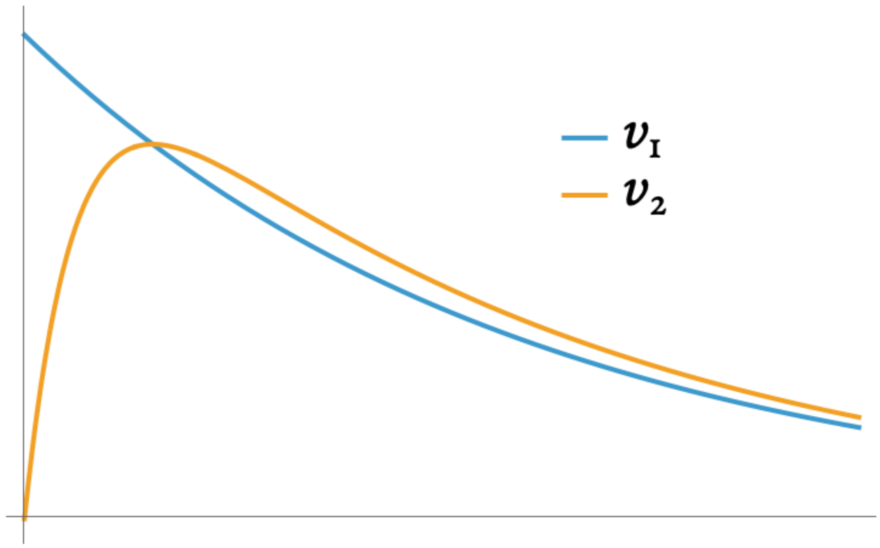


Figure 2: Velocities v_1 and v_2 plotted against time

Solution

Our physics intuition suggests that there will be some kind of 'lag' between the motion of the two masses. If we start out by moving only m_1, then m_2 won't immediately reach the same speed, and perhaps never will. However, if the speed is reasonably small, we do expect m_2 to be carried along with m_1 eventually. We know that sometimes it's possible to 'pull the rug from underneath' by moving something very quickly.
It seems that this all depends very much on how much friction there is between m_1 and m_2. It's unclear at this stage what this dependence is, exactly.

2. Using Figure 2, make a sketch of x_1 and x_2 as a function of time for the first few moments of the motion, until the orange and blue lines shown above intersect. Start both x_1 and x_2 from zero, which effectively means we're re-defining zero to be wherever things started. Your knowledge of calculus will help you with this task. Remember that this is a qualitative, not a quantitative task.

Solution

For this solution, I have provided a plot rather than a sketch. It is possible to arrive at a sketch using the following arguments.
Figure 2 tells us that the speed of mass 1 is initially some large number (not infinite), while the speed of mass 2 is initially zero or close to zero. Thus, the slope of the position-vs-time graphs should start off as 'some positive number' for m_1 and 'basically zero' for m_2.
Figure 2 also tells us that the speed of m_1 decreases while the speed of m_2 increases. Thus, the slope of the two graphs should change accordingly.
At the point where the orange and blue lines intersect, we cannot say that the positions are the same. In fact, we **know** that mass 1 has moved much more than mass 2 by that point because the area under the graph for mass 2 is clearly lower than the area under the graph for mass 1. We can say that the slopes of the two graphs should be the same at that point.

All these characteristics are shown in the sketch below in Figure 4.

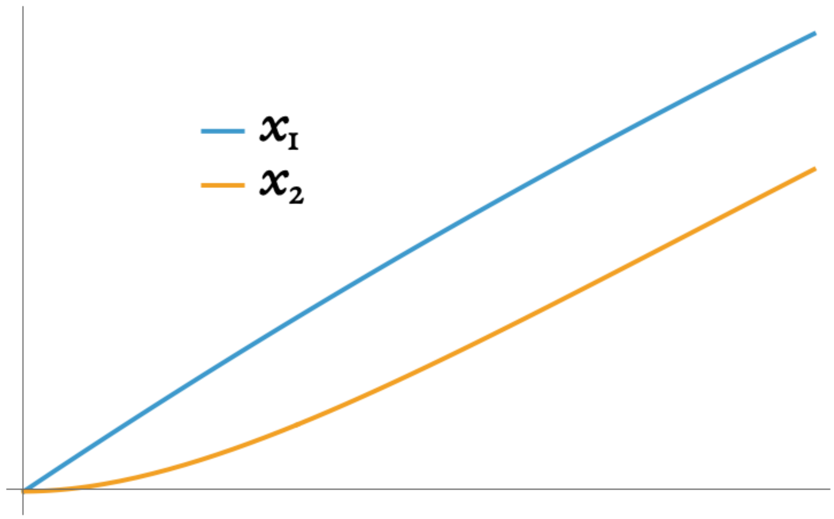


Figure 3: Positions x_1 and x_2, shifted so they both start from zero, plotted against time

3. Also make a sketch of $(x_2 - x_1)$ against time on the same set of axes as above, and explain what this represents physically.

Position vs. velocity

The point where v_1 and v_2 intersect is where the two boxes have the same velocity, not position!

Solution

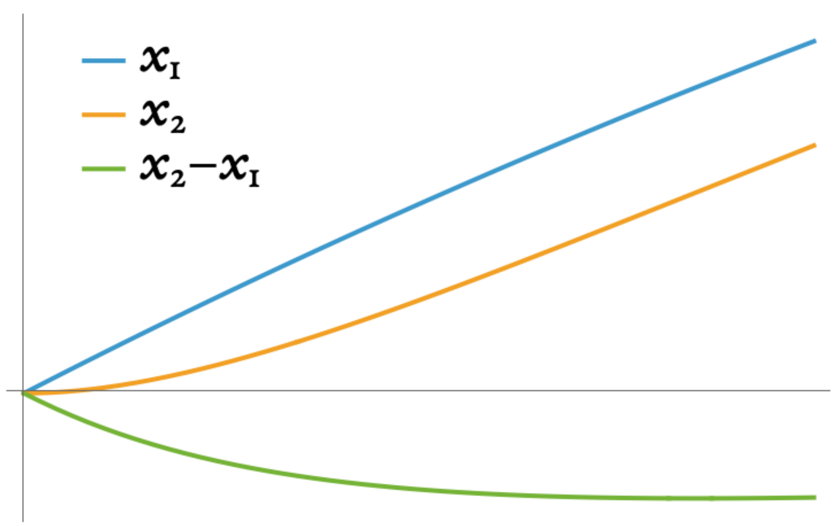


Figure 4: Positions x_1 and x_2, shifted so they both start from zero, plotted against time

The fact that this is negative shows that mass 2 is actually moving backward relative to mass 1. It's being 'left behind', but it's catching up to some extent.
For a finite sized lower object, this would mean that the object on top falls off.