

Problem Set 11 Solutions

ENGR 12, Spring 2026.

Solutions

1 Frequency Response of an underdamped system close to resonance

Consider a second-order system subjected to a forcing input function

$$m\ddot{x} + b\dot{x} + kx = f(t) \quad (1)$$

with $m = 1, b = 2, k = 5$.

1.1 Roots

Determine the roots of the characteristic polynomial and plot them in the complex plane. Is the system stable or unstable, and underdamped or overdamped?

Solution


The characteristic polynomial is $ms^2 + bs + k$ and its roots are given by the quadratic formula. We find that the roots are

$$-1 \pm 2i$$

So this system is underdamped (roots have nonzero imaginary part) and stable (real part is negative).

1.2 Fill in the following table for this system.


Quantity	Value
ω_n	
ζ	
ω_d	
ω_r	

 Solution

Quantity	Value
ω_n	$\sqrt{5} \approx 2.236$
ζ	$1/\sqrt{5} \approx 0.447$
ω_d	2
ω_r	$\sqrt{3} \approx 1.732$

1.3 Transfer Function

Write down an expression in terms of s for the transfer function of this system. For the purpose of this transfer function, let the output be kx instead of x , and let the input be f as usual, as was done in [lecture 22](#).

 Solution

Defining the transfer function to be $\frac{kX(s)}{F(s)}$, we can use the Laplace Transform on the given Equation 1:

$$ms^2X(s) + bsX(s) + kX(s) = F(s)$$

The output is $kX(s)$ and the input is $F(s)$. First, let's find an equation for $X(s)$:

$$X(s) = \frac{F(s)}{ms^2 + bs + k}$$

Now, the transfer function is $kX(s)$ divided by $F(s)$. So we can write

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k} \implies \frac{kX(s)}{F(s)} = \frac{k}{ms^2 + bs + k}$$

So the desired transfer function relating the output $kX(s)$ to the input $F(s)$ is

$$\boxed{\frac{kX(s)}{F(s)} = \frac{k}{ms^2 + bs + k} = \frac{5}{s^2 + 2s + 5}} \quad (2)$$

1.4 Bode Plot

1.4.1 Expression for amplitude ratio

Use the expressions developed in [Lecture 22](#) to calculate an expression for the quantity that will be placed on the vertical axis of the magnitude Bode plot for this system. Your expression should be in terms of **only** the variable r , as defined in lecture.

 Solution

The desired quantity is $|T(i\omega)|$ where $T(s)$ is the transfer function. In [Lecture 22](#), we found that in terms of r this quantity is equal to

$$\frac{1}{\sqrt{(1-r^2)^2 + 4\zeta^2 r^2}}$$

All that's left to do is to calculate $\zeta = b/\sqrt{4mk} = \frac{1}{\sqrt{5}}$, which means that the quantity to be plotted on the Bode plot's magnitude axis is

$$\boxed{\frac{1}{\sqrt{(1-r^2)^2 + (4/5)r^2}}}$$

1.4.2 Expression for phase shift

Use the expressions developed in [Lecture 22](#) to calculate an expression for the quantity that will be placed on the vertical axis of the phase Bode plot for this system. Your expression should be in terms of **only** the variable r , as defined in lecture.

 Solution

This problem asks us to find the argument of the complex number $T(i\omega)$ when expressed in terms of r . The complex number in question is

$$\frac{1}{1-r^2 + 2\zeta r i}$$

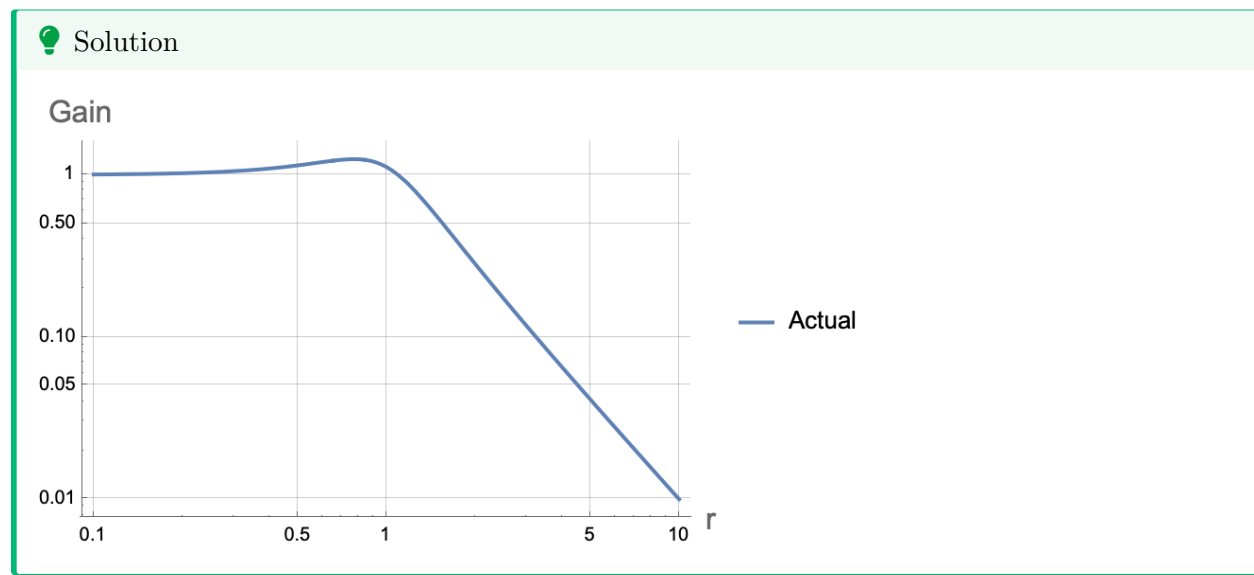
The argument of this complex number is the argument of the numerator minus the argument of the denominator, because $\frac{r_1 e^{i\phi_1}}{r_2 e^{i\phi_2}} = \frac{r_1}{r_2} e^{i(\phi_1 - \phi_2)}$.

The argument of the numerator is zero, because real numbers are on the positive real axis and have argument zero. The argument of the denominator is $\tan^{-1} \frac{2\zeta r}{1-r^2}$. So the expression we are looking for is

$$\boxed{-\tan^{-1} \frac{2\zeta r}{1-r^2}}$$

1.4.3 Plot the plot

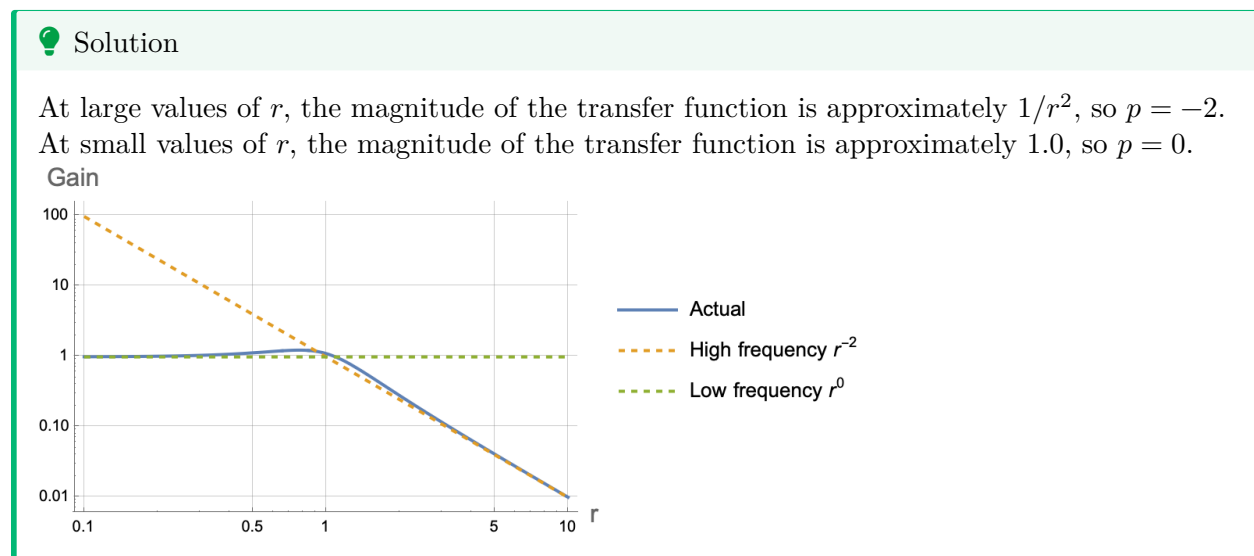
Use a computer program to make the (magnitude) Bode plot for this system, with $0.1 < r < 10$ on the horizontal axis. Make sure to use a log-log scale for your plot.



1.4.4 Estimate low frequency and high-frequency behavior

At very high input frequencies, the system's (magnitude) Bode plot can be approximated by a straight line on a log-log plot. The equation of this 'straight line' is of the form r^p where p is a number to be found. Sketch (or plot) this straight line on top of your answer to Section 1.4.3 and determine the value of p .

At very low input frequencies, the system has a gain of 1. Also sketch (or plot) a straight-line approximation to this system's Bode plot valid at these low frequencies, on top of your answer to Section 1.4.3.



2 Interpreting Bode Plots

2.1 Fourth-order

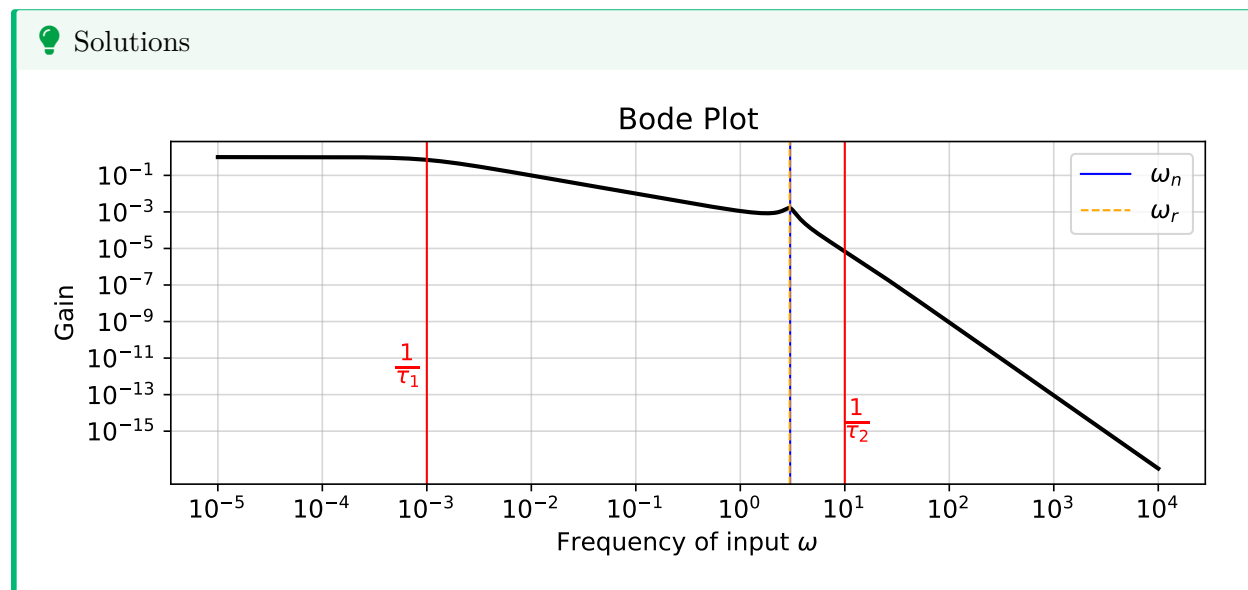
Consider the following Bode plot, which describes a system governed by the transfer function

$$T(s) = \frac{9}{(s^2 + 0.6s + 9)(0.1s + 1)(1000s + 1)}$$

Add a vertical line corresponding to:

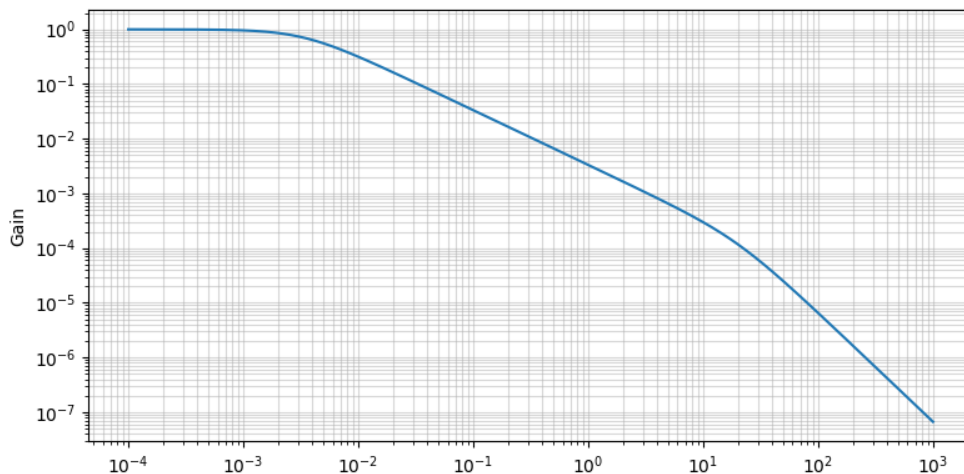
- Each of the two ‘corner frequencies’ attributable to the real poles of this transfer function,
- The undamped natural frequency of this system, and
- The resonant frequency of this system.

There should be 4 lines in total.

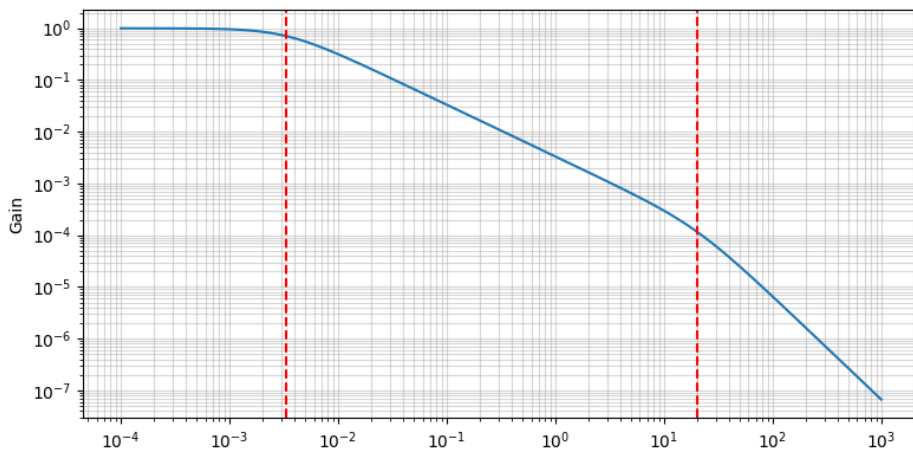


2.2 Second-order

As closely as you can, determine the transfer function whose Bode plot is shown below.



💡 Solution



By observing the Bode plot closely, we can tell that it arises from the following transfer function:

$$\frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

where $\tau_1 \approx 0.05$ and $\tau_2 \approx 300$.

3 Another second-order system

Use the values $m = 5, k = 10, b = 4, \omega = 4$.

Consider a second-order system subjected to a unit-amplitude sinusoidal input,

$$m\ddot{x} + b\dot{x} + kx = \sin \omega t$$

The response of the system, $x(t)$, is shown below.

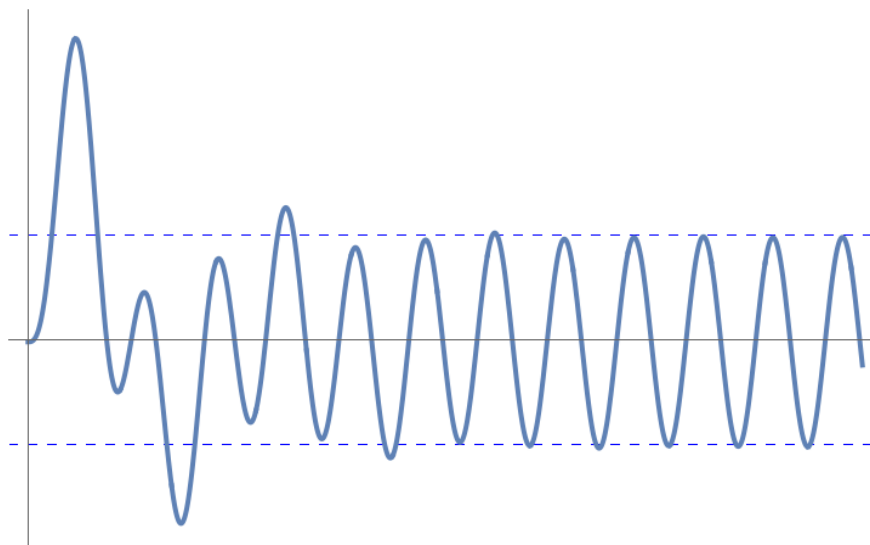


Figure 1

3.1 Amplitude ratio

Determine the amplitude of the sinusoidal curve in the right half of the graph above. Give your answer as a number.

💡 Solution

This problem can be solved using the magnitude part of the Bode plot for a second-order system with the parameter values given.

Since $m = 5, k = 10, b = 4$, we have the natural frequency $\omega_n = \sqrt{10/5} = \sqrt{2}$. The damping ratio is $4/\sqrt{4 \times 5 \times 10} = \sqrt{2}/5$. The ratio of the applied frequency $\omega = 4$ to the undamped natural frequency $\omega_n = \sqrt{2}$ is $r = 4/\sqrt{2}$.

We know that the magnitude ratio is

$$\frac{1}{\sqrt{(1-r^2)^2 + 4\zeta^2 r^2}}$$

Substituting all the relevant values, we find that the amplitude ratio is ≈ 0.139 .

3.2 Transient vs. steady-state part of the frequency response

Write down the **transient part** of the response of this system to the given forcing (input) function and the **steady-state part** of the response of this system to the given forcing (input) function.

Both parts should be functions of time involving no symbols other than the common mathematical functions and constants like \sin , π and e . Adding them up should give the graph in Figure 1.

Solution

To solve this problem, we first note that the response of a system with transfer function $\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ to a sinusoidal input with amplitude ω is, in the frequency domain,

$$\frac{\omega}{s^2 + \omega^2} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (3)$$

From the given numbers in this problem, we can write the above numerically as

$$\frac{8}{(s^2 + 16)(s^2 + \frac{4}{5}s + 2)}$$

We know that the end result must involve a shifted sine in the time domain, which is equivalent to some combination of sines and cosines of the same frequency as the input frequency. This gives us the hint that we should break apart Equation 3 into some parts, two of which should be a sine and a cosine.

The other two parts should be some sort of transient term, which — from what we know about underdamped second-order systems — have the form $e^{-at} \sin bt$ and/or $e^{-at} \cos bt$. Thus, we need to use partial fractions to write an equation of the form

$$A \frac{4}{s^2 + 16} + B \frac{s}{s^2 + 16} + C \frac{s + 2/5}{(s + 2/5)^2 + 46/25} + D \frac{\sqrt{46/25}}{(s + 2/5)^2 + 46/25} \quad (4)$$

Now, how did we know what form the third and fourth terms in the above equation take? The answer is that we use the ‘completed square’ form of $s^2 + 4s/5 + 2$, i.e.,

$$s^2 + 4s/5 + 2 = \left(s + \frac{2}{5}\right)^2 + \frac{46}{25}$$

Looking closely at the Laplace Transforms table, this tells us that we should look for exponentially decaying sines and cosines $e^{-at} \cos bt$ and $e^{-at} \sin bt$ where $a = 2/5$ and $b^2 = 46/25$.

We then multiply through the terms in Equation 4 by the right quantities so that we get a single large fraction with lots of terms, all of which must be identical to the original Equation 3. This gives us

$$\frac{4A(s^2 + \frac{4}{5}s + 2)}{(s^2 + 16)(s^2 + \frac{4}{5}s + 2)} + \frac{Bs(s^2 + \frac{4}{5}s + 2)}{(s^2 + 16)(s^2 + \frac{4}{5}s + 2)} + \frac{C(s + 2/5)(s^2 + 16)}{(s^2 + 16)(s^2 + \frac{4}{5}s + 2)} + \frac{D\sqrt{46/25} \cdot (s^2 + 16)}{(s^2 + 16)(s^2 + \frac{4}{5}s + 2)}$$

The denominators are now the same, and the numerator should be equal to 8. Notice that the numerator is a cubic expression in s , so we can write four separate equations for the four unknowns: one using the coefficients of s^3 , one using the coefficients of s^2 , one using the coefficients of s^1 , and one using the coefficients of s^0 . The four equations are:

$$\begin{aligned}
 B + C &= 0 \\
 4A + 4B/5 + 2C/5 + \sqrt{46/25}D &= 0 \\
 16A/5 + 2B + 16C &= 0 \\
 16D\sqrt{46/25} + 8A + 32C/5 &= 0
 \end{aligned}$$

At this stage, it is okay to use a computer program to solve the above system of simultaneous equations. Using Mathematica, I found the solutions to be

$$A = -\frac{175}{1289}, \quad B = -\frac{40}{1289}, \quad C = \frac{40}{1289}, \quad D = \frac{1790\sqrt{2/23}}{1289}$$

Now, the purpose of these four terms was to act as coefficients of the different terms in the solution. Let's recall:

$$\underbrace{A \frac{4}{s^2 + 16}}_{\mathcal{L}[\sin 4t]} + \underbrace{B \frac{s}{s^2 + 16}}_{\mathcal{L}[\cos 4t]} + \underbrace{C \frac{s + 2/5}{(s + 2/5)^2 + 46/25}}_{\mathcal{L}[e^{-2t/5} \cos(\sqrt{46/25}t)]} + \underbrace{D \frac{\sqrt{46/25}}{(s + 2/5)^2 + 46/25}}_{\mathcal{L}[e^{-2t/5} \sin(\sqrt{46/25}t)]}$$

So the time domain solution is

$$A \sin 4t + B \cos 4t + C e^{-2t/5} \cos(\sqrt{46/25}t) + D e^{-2t/5} \sin(\sqrt{46/25}t)$$

The first two terms are the steady-state solution. It is

$$-\frac{175}{1289} \sin 4t - \frac{40}{1289} \cos 4t.$$

Notice that this is a periodic function whose amplitude is given by $\sqrt{A^2 + B^2} \approx 0.139$, which is what the Bode plot formula from Section 3.1 predicted.

The last two terms are the transient solution. Thus, the transient solution is

$$\begin{aligned}
 e^{-2t/5} \left(\frac{40}{1289} \cos(\sqrt{46/25}t) + \frac{1790\sqrt{2/23}}{1289} \sin(\sqrt{46/25}t) \right) \\
 \approx 0.031 e^{-0.4t} \cos 1.356t + 0.409 \sin 1.356t
 \end{aligned}$$

