

Problem Set 12 Solutions

ENGR 12, Spring 2026.

Solutions

1 Moving a system of two masses

1.1 Derive the state-space equations

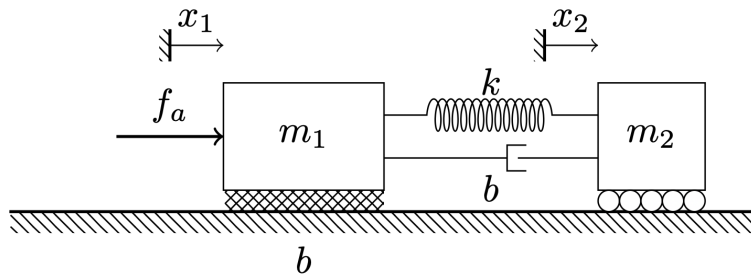


Figure 1

Derive the state-space equations for the above system, considering $f_a(t)$ to be the sole input and $x_2 - x_1$ to be the sole output. Let $m_1 = m_2 = m$. Use the state variables

1. \dot{x}_1
2. \dot{x}_2
3. $x_2 - x_1$

💡 Solutions

Using the following FBD diagram,

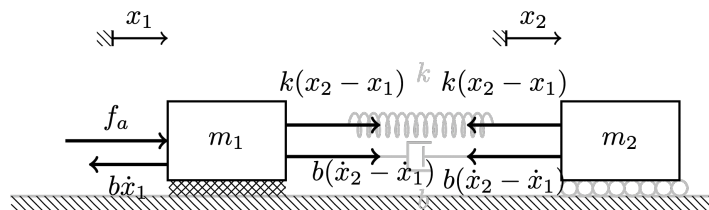


Figure 2

we can write Newton's second law for the two masses separately.

1. For mass 1, we have

$$m\ddot{x}_1 = k(x_2 - x_1) + b(\dot{x}_2 - \dot{x}_1) + f_a - b\dot{x}_1$$

2. For mass 2, we have

$$m\ddot{x}_2 = -k(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1)$$

The state-space equations are

$$\frac{d}{dt} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ z \end{bmatrix} = \begin{bmatrix} -\frac{2b}{m} & \frac{b}{m} & \frac{k}{m} \\ \frac{b}{m} & -\frac{b}{m} & -\frac{k}{m} \\ -1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ z \end{bmatrix} + \begin{bmatrix} 1/m \\ 0 \\ 0 \end{bmatrix} \cdot [f_a]$$

$$y = Cx + Du$$

where y is the output vector. If the only output is the third state variable, $z = x_2 - x_1$, the matrix $C = [0, 0, 1]$ and the matrix $D = [0]$.

1.2 Explore in MATLAB

Note

For the entirety of Section 1.2, the physical system is identical to Figure 2, but the outputs we are interested in are different from the output in Section 1.1. We are now interested in the following two outputs:

1. \dot{x}_1
2. \dot{x}_2

Use the following parameter values: $b = 5$, $m = 1$ and $k = 3$.

Hint

Consider which of the matrices A, B, C and D need to change before you can use MATLAB. Also check out [this slide](#) to make sure the dimensions of your matrices are correct.

1.2.1 Transfer Functions using MATLAB

Use the `ss` and `tf` functions to determine the transfer function between the input f_a and the two outputs:

1. \dot{x}_1
2. \dot{x}_2

Give your answer in the form of a rational function of s , the frequency variable.

Solutions

The following MATLAB code was used.

```
b = 5;
m = 1;
k = 3;

A = [-2*b/m, b/m, k/m;
     b/m, -b/m, -k/m;
     -1, 1, 0];

B = [1/m;
     0;
     0];

% Output = the first and second state variables
C = [1,0,0;
     0,1,0];

D = [0;
     0];

sys1 = ss(A, B, C, D);
tfsys1 = tf(sys1)
```

This gave the following output:

```
From input to output...
          s^2 + 5 s + 3
1:  -----
    s^3 + 15 s^2 + 31 s + 15

          5 s + 3
2:  -----
    s^3 + 15 s^2 + 31 s + 15
```

i.e., the desired transfer functions are:

1. From input f_a to output \dot{x}_1 :

$$\frac{s^2 + 5s + 3}{s^3 + 15s^2 + 31s + 15}$$

2. From input f_a to output \dot{x}_2 :

$$\frac{5s + 3}{s^3 + 15s^2 + 31s + 15}$$

1.2.2 Poles in MATLAB

You will notice that the transfer functions you have found in Section 1.2.1 have third-degree polynomials in s in their denominator. This tells us that the system is third order, and there should be three poles.

Tip

Recall that a **pole** is a value of s that would make the transfer function undefined. A pole of a transfer function is also a root of the **denominator** polynomial of that transfer function.

Determine the numerical value of the three poles of this system. To do this, you may either use MATLAB's `pole` function, or use a computer algebra program such as WolframAlpha to find the roots of the cubic polynomial in the denominator.

Give your answer as a set of three (possibly complex) numbers.

Solutions

Following on from the last part's MATLAB code, we can use

```
>> pole(tfsys1)

ans =

    -12.6416
     -1.6307
     -0.7276
```

So the three poles are $-12.6, -1.63, -0.73$.

1.2.3 Zeros in MATLAB

A transfer function has zeros in addition to poles.

Tip

Recall that a **zero** of a transfer function is a value of s that would make the transfer function equal to zero. A zero of a transfer function is also a root of the **numerator** polynomial of that transfer function.

Find the zeros of each of the transfer functions that you found in Section 1.2.1. You can either do this mathematically or you can use the `zero` function of MATLAB.

i Note

The MATLAB function `zero` only works for ‘single-input, single-output’ systems, not for multiple-input or multiple-output systems. However, it is still possible to use `zero` here by extracting two separate transfer functions out of the output of `tf` from your answer to Section 1.2.1.

💡 Solutions

Following on from the last part’s MATLAB code, we can use

```
>> zero(tfsys1(1))
```

```
ans =
```

```
-4.3028
```

```
-0.6972
```

```
>> zero(tfsys1(2))
```

```
ans =
```

```
-0.6000
```

So the zeros are:

1. For the output \dot{x}_1 : $s = -4.30$, $s = -0.70$
2. For the output \dot{x}_2 : $s = -3/5$

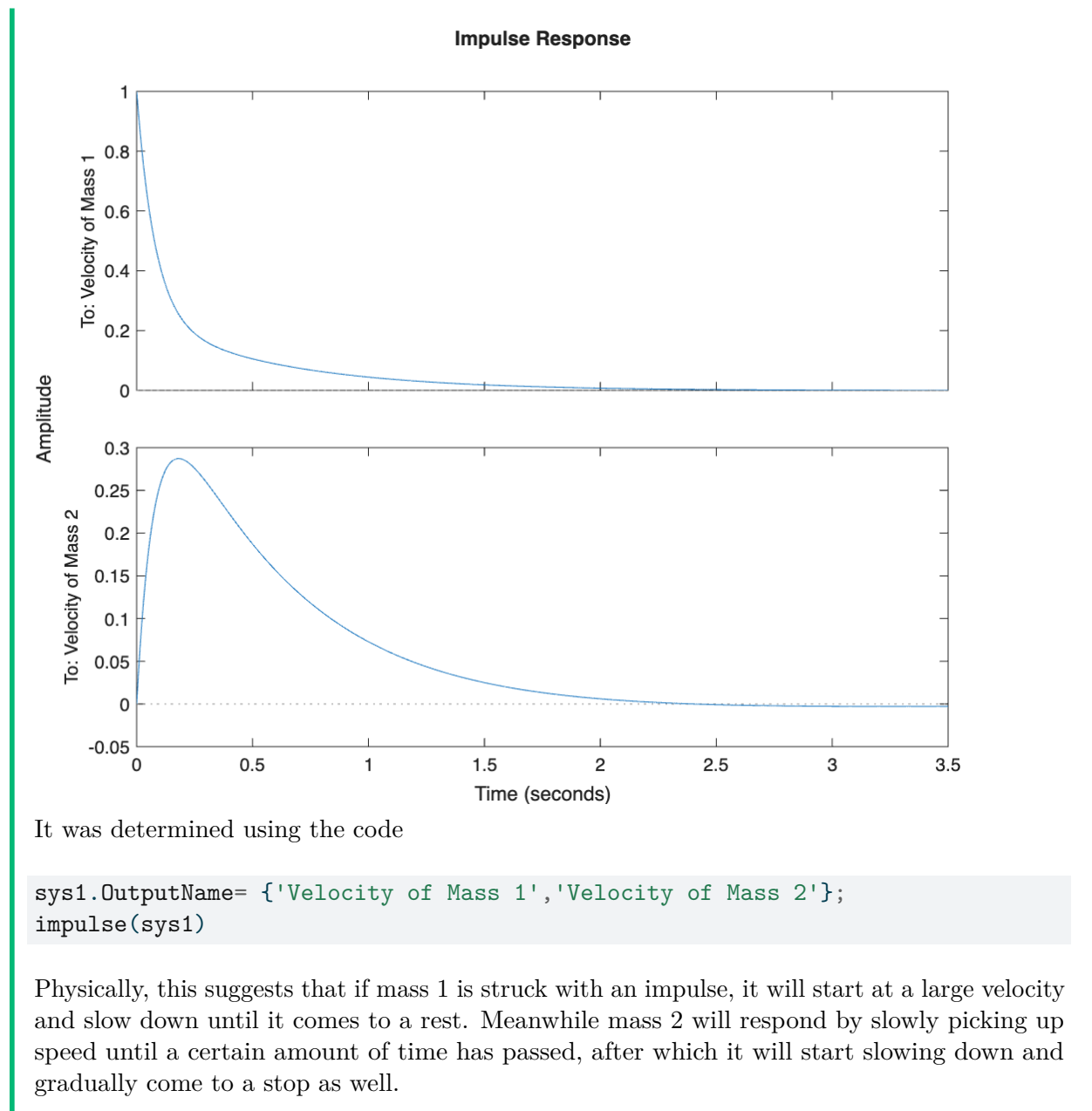
1.2.4 Impulse Response

We would like to examine the response of this system to an **impulse input** where $f_a(t)$ takes the form of a Dirac Delta function. Use MATLAB’s `impz` function to **produce a time-domain plot** of how \dot{x}_1 and \dot{x}_2 respond to such an input. It is okay to directly reproduce MATLAB’s plots without any modification.

Also write a brief description of what this represents physically.

💡 Solutions

The impulse response is as follows.



1.3 Repeat with a smaller b value

Repeat Section 1.2.1, Section 1.2.2, Section 1.2.3, and Section 1.2.4 for the following system parameters, and briefly comment on the differences you observe (one to two sentences).

$$m = 1, k = 3, b = 0.2.$$

 Solutions

Transfer functions

From input to output...

$$1: \frac{s^2 + 0.2s + 3}{s^3 + 0.6s^2 + 6.04s + 0.6}$$

$$2: \frac{0.2s + 3}{s^3 + 0.6s^2 + 6.04s + 0.6}$$

1. For \dot{x}_1 :

$$\frac{s^2 + 0.2s + 3}{s^3 + 0.6s^2 + 6.04s + 0.6} \quad (1)$$

2. For \dot{x}_2 :

$$\frac{0.2s + 3}{s^3 + 0.6s^2 + 6.04s + 0.6} \quad (2)$$

Poles

The poles of this system are located at

$$\begin{aligned} &-0.2499 + 2.4346i \\ &-0.2499 - 2.4346i \\ &-0.1002 + 0.0000i \end{aligned}$$

Zeros

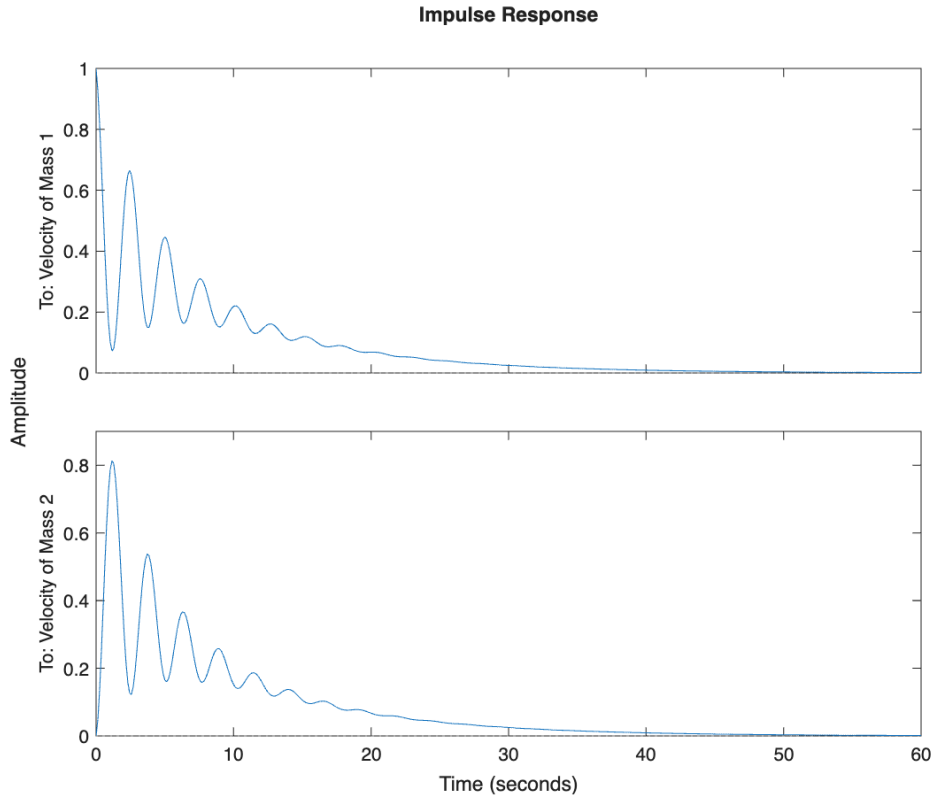
The zeros of Equation 1 are located at

$$\begin{aligned} &-0.1000 + 1.7292i \\ &-0.1000 - 1.7292i \end{aligned}$$

There is only one zero of Equation 2 and it is located at

$$-15.000$$

Impulse Response

**Comment**

We notice that the system now has complex poles, and there is oscillation consistent with 'underdamping'. This suggests that the earlier value of $b = 5$ was overdamped, whereas $b = 0.2$ is underdamped.

2 The shape of matrices in state-space equations

Write out a sample set of matrices and vectors (the entries are irrelevant; the size and shape of the matrices are what you should pay attention to) for the state-space equations when a linear physical system has:

1. Three state variables, 3 outputs, and 2 inputs.
2. Two state variables, 1 input, and 3 outputs.
3. Two state variables, 1 output, and 3 inputs.
4. Four state variables, 2 inputs, and 4 outputs that are identical to the state variables. (for this one, your matrix C and D must have the numerically correct entries)

For each part, write down:

- The matrices A , B , C and D with the correct shape and dummy entries.
- The vectors u , x and y with entries labeled with $u_1, u_2, \dots, x_1, x_2, \dots$, and y_1, y_2, \dots
- The state-space equation in full form.

Solutions

2.1 Three state variables, 3 outputs, 2 inputs

1. The matrices are:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}, \quad D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \\ D_{31} & D_{32} \end{bmatrix}$$

2. The vectors are:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

3. The equations can be written as:

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} &= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \\ D_{31} & D_{32} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned}$$

2.2 Two state variables, 1 input and 3 outputs

1. The matrices are:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix}, \quad D = \begin{bmatrix} D_{11} \\ D_{21} \\ D_{31} \end{bmatrix}$$

2. The vectors are:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad u = [u_1], \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

3. The equations can be written as:

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} \cdot [u_1] \\ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} &= \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} D_{11} \\ D_{21} \\ D_{31} \end{bmatrix} \cdot [u_1] \end{aligned}$$

2.3 Two state variables, 1 output and 3 inputs

1. The matrices are:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix}, \quad C = [C_{11} \quad C_{12}], \quad D = [D_{11} \quad D_{12} \quad D_{13}]$$

2. The vectors are:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad y = [y_1]$$

3. The equations can be written as:

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} \cdot [u_1] \\ [y_1] &= [C_{11} \quad C_{12}] \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [D_{11} \quad D_{12} \quad D_{13}] \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \end{aligned}$$

2.4 Four state variables, 2 inputs and 4 outputs

Where the outputs are just the state variables.

1. The matrices are:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \\ B_{41} & B_{42} \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix}, \quad D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \\ D_{31} & D_{32} \\ D_{41} & D_{42} \end{bmatrix}$$

Where we know that

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2. The vectors are:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

3. The equations can be written as:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \\ B_{41} & B_{42} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

3 A vibration isolation system

Consider the following system, which is a simplified representation of a vibration isolation apparatus.

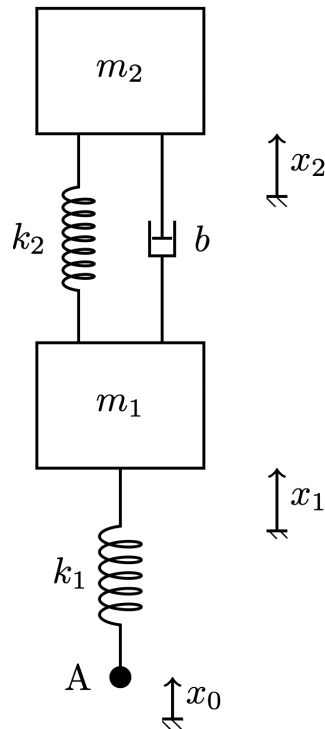


Figure 3

Assume that x_1 and x_2 and x_0 have been defined such that when all of them are zero, there is no force in any of the springs. The point A is ‘massless point’ that has position x_0 but no inertia.

💡 Tip

In this problem, you do not need a free-body diagram for A

We will consider this system with one input, $x_0(t)$, and two outputs that are of interest to us from an engineering perspective:

1. The total force experienced by m_2
2. The position of mass m_2 .

3.1 Deriving the state-space equations

Determine the state-space equations for this system by writing out the matrices and vectors. The state variables you should use are:

1. x_1
2. \dot{x}_1

3. x_2
4. \dot{x}_2

💡 Solutions

First, let's label the forces on this diagram, so that a FBD for each mass can be assembled.

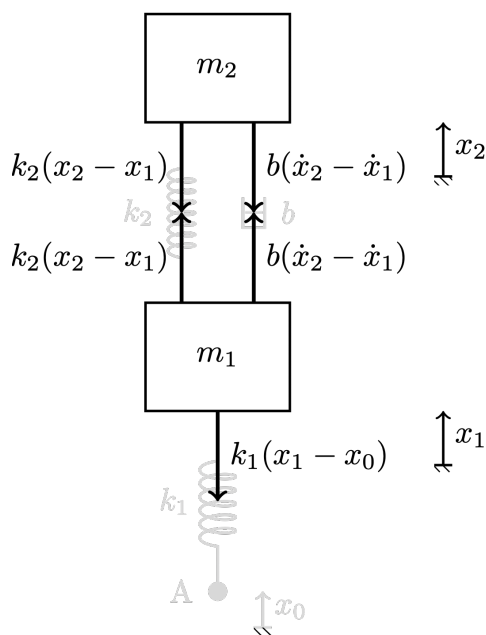


Figure 4

From the free body diagrams for this system we can write the following two equations representing Newton's second law:

1. From the FBD of mass 1:

$$m_1 \ddot{x}_1 = k_2(x_2 - x_1) + b(\dot{x}_2 - \dot{x}_1) - k_1(x_1 - x_0)$$

2. From the FBD of mass 2:

$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1)$$

Rearranging this to get derivatives of the state variables, we can write

$$\ddot{x}_1 = \frac{k_2}{m_1}(x_2 - x_1) + \frac{b}{m_1}(\dot{x}_2 - \dot{x}_1) - \frac{k_1}{m_1}(x_1 - x_0)$$

$$\ddot{x}_2 = -\frac{k_2}{m_2}(x_2 - x_1) - \frac{b}{m_2}(\dot{x}_2 - \dot{x}_1)$$

Which then leads us to the **state-variable equation**

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} - \frac{k_2}{m_1} & -\frac{b}{m_1} & \frac{k_2}{m_1} & \frac{b}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{b}{m_2} & -\frac{k_2}{m_2} & -\frac{b}{m_2} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k_1}{m_1} \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x_0 \end{bmatrix}$$

The output equation should give us the two outputs as a linear combination of the state variables and the inputs, if applicable. In this case, the output equation is

$$y = \begin{bmatrix} k_2 & b & -k_2 & -b \\ -1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x_0 \end{bmatrix}$$

3.2 Step and Impulse Response

Use MATLAB to make plots of the step and impulse response of this system. Comment on the graphs by writing a short (two-to-three sentence) description of what the graphs show by referring to Figure 4. Use the following values for your parameters.

```
m1 = 10;
m2 = 4;
k1 = 5;
k2 = 5;
b = 10;
```

Solutions

The following MATLAB code was used.

```

m1 = 10;
m2 = 4;
k1 = 5;
k2 = 5;
b = 10;

A = [0,          1,      0,  0;
     -k1/m1-k2/m1, -b/m1, k2/m1, b/m1;
     0,          0,      0,  1;
     k2/m2,      b/m2, -k2/m2, -b/m2];

B = [0;
     k1/m1;
     0;
     0];

C = [k2, b, -k2, -b;
     0,  0,  1,  0];

D = [0;
     0];

sys=ss(A,B,C,D);

sys.OutputName = {'Force on m2','Position of m1'};

figure(1); clf;
step(sys);
saveas(gcf,"vibisol-stepresponse.png");
clf;
impulse(sys);
saveas(gcf,"vibisol-impulseresponse.png");

```

3.2.1 Step Response

The step response refers to how this system behaves if the massless point A is suddenly moved upward and kept there. From MATLAB, we find the step response to be the following.

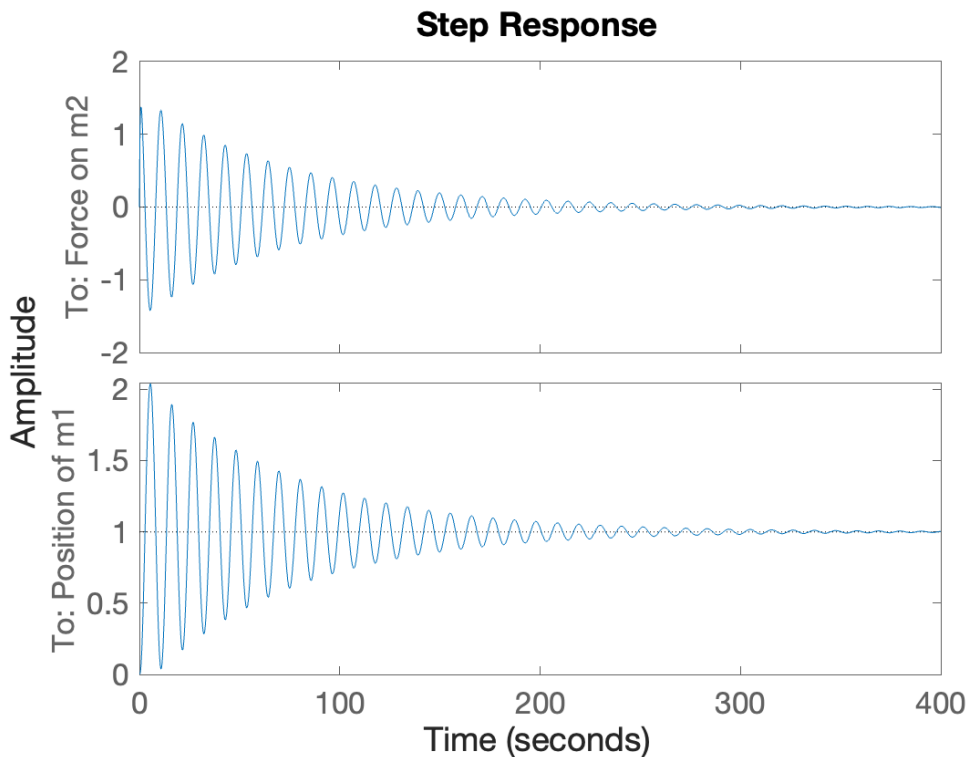


Figure 5

We see that the position of m_1 moves up, oscillates about 1, and then settles down to 1. Since this is a **unit** step input, this means that the mass m_1 moves up, in the end, just as much as we move the point A.

The force on m_2 experiences some oscillations as the spring k_2 and the damper b exert forces on it, but eventually these forces die out for a total force of zero.

3.2.2 Impulse Response

The impulse response refers to how this system behaves if the massless point A is suddenly moved upward and then brought back down.

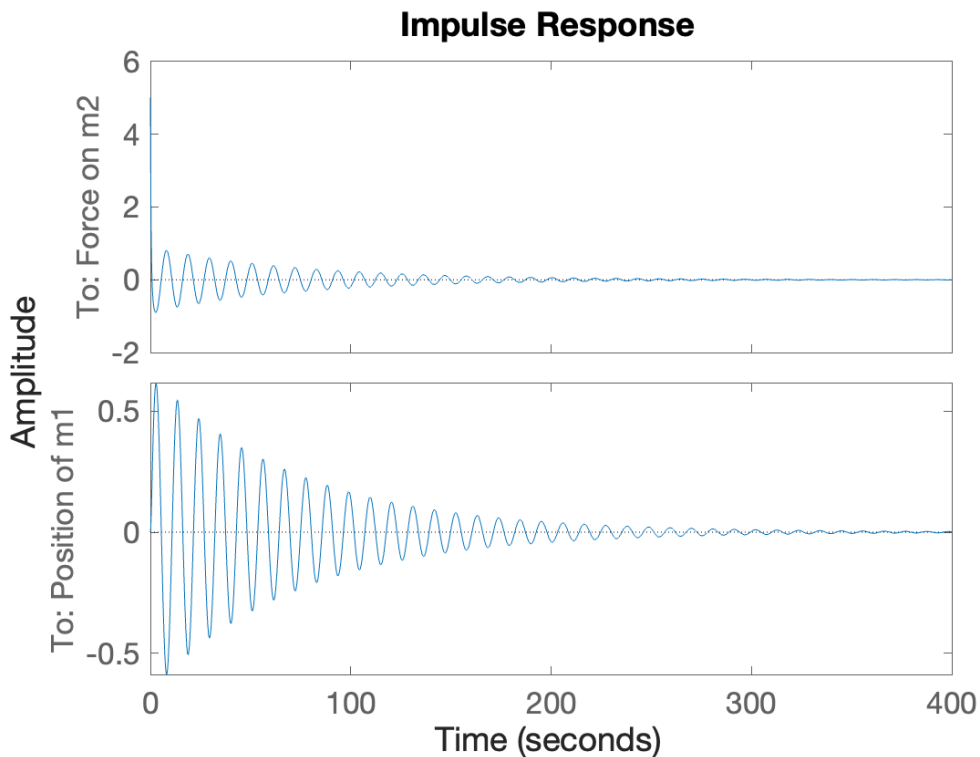


Figure 6

We see that the position of m_1 moves up suddenly, but then comes back down to zero. The force on m_2 experiences significantly more oscillations than in the step response case, but once again the force dies down to zero eventually.