

Problem Set 13 Solutions

ENGR 12, Spring 2026.

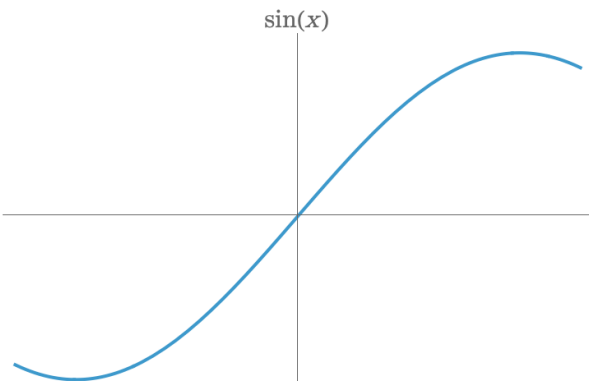
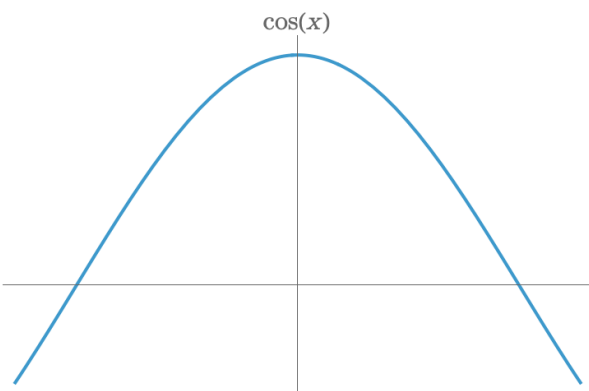
Solutions

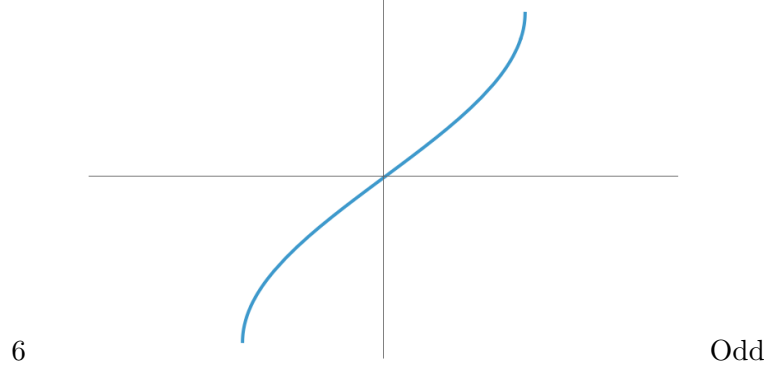
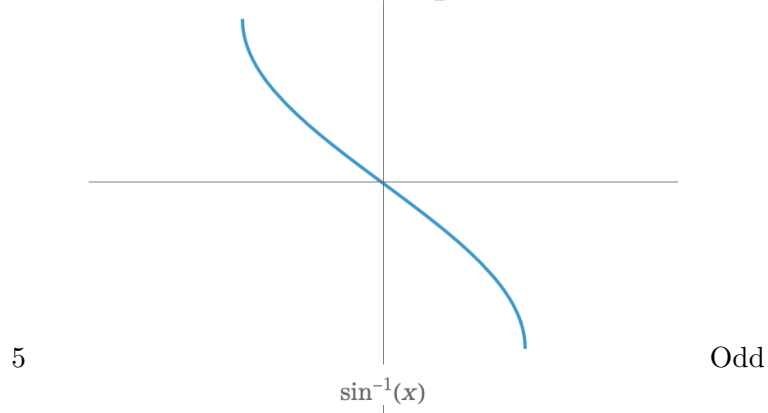
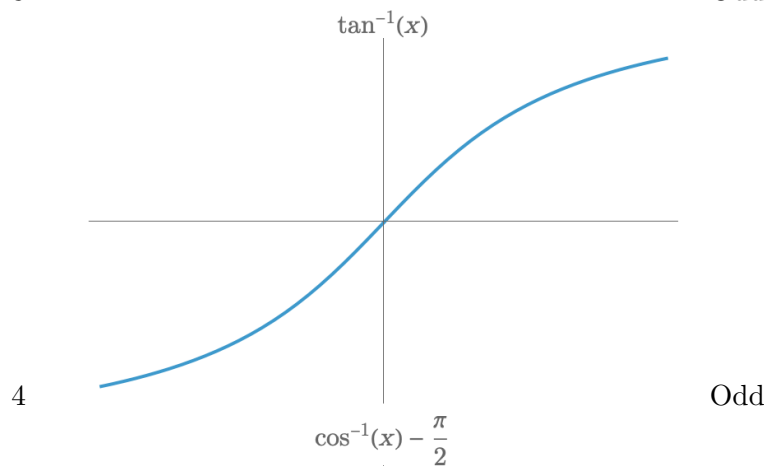
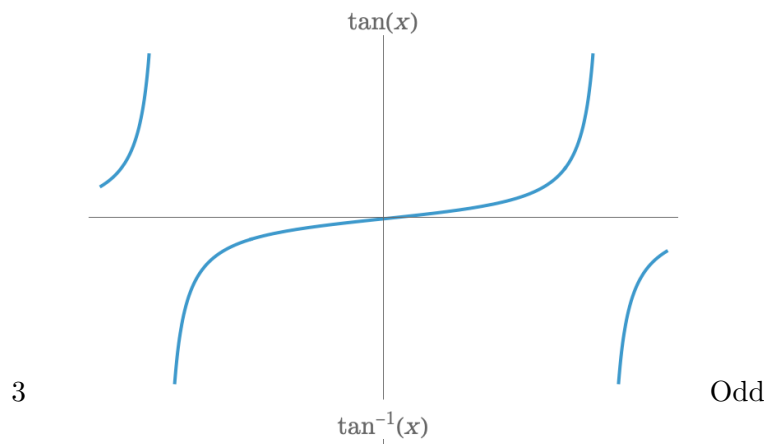
1 Even and odd functions

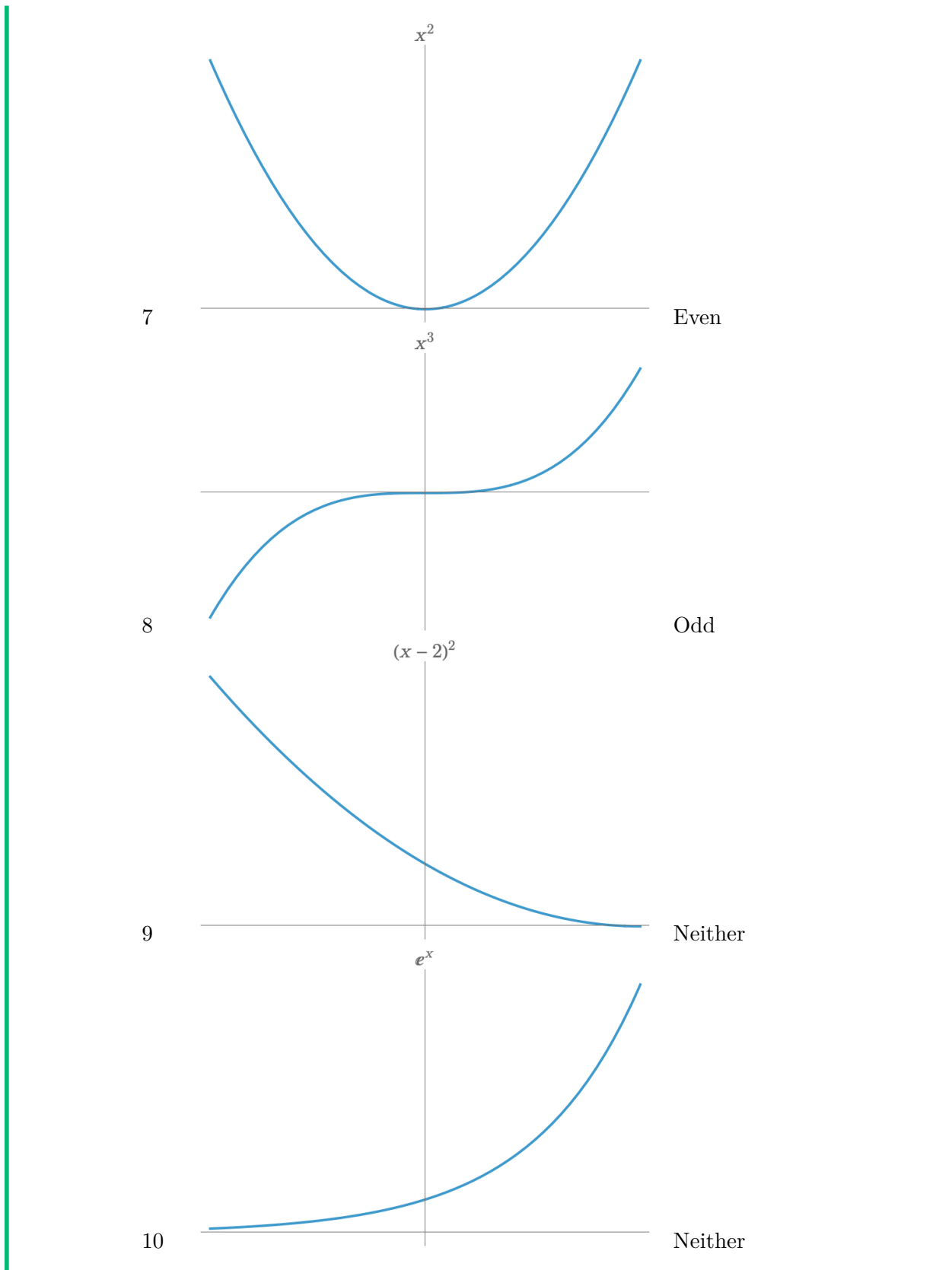
Sketch (or plot) the following functions and determine whether they are even, odd, or neither. Use radians instead of degrees, if relevant.

1. $\sin x$
2. $\cos x$
3. $\tan x$
4. $\tan^{-1} x$
5. $\cos^{-1} x - \pi/2$ over the domain $(-1, +1)$
6. $\sin^{-1} x$
7. x^2
8. x^3
9. $(x - 2)^2$
10. e^x

 Solution

No.	Graph	Characteristic
1		Odd
2		Even





2 Some Rules about even and odd functions

Complete the following statements and give an example in each case. Use the words **even**, **odd**, or **neither** to fill in each blank.

Statement

An even function times an even function is **even**

An even function plus an even function is ?

An even function times an odd function is ?

An even function plus an odd function is ?

An odd function multiplied by an odd function is ?

An odd function plus an odd function is ?

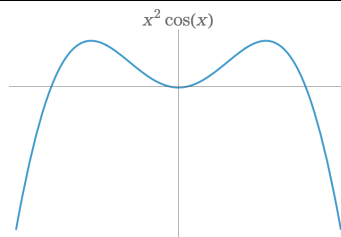
By using candidate functions like $\sin x$, $\cos x$, x^2 , and x , illustrate each of the above statements with an example by checking for yourself whether the result is even, odd, or neither even nor odd. Your example should consist of a plot or sketch showing clearly which two functions were multiplied or added to each other.

Solution

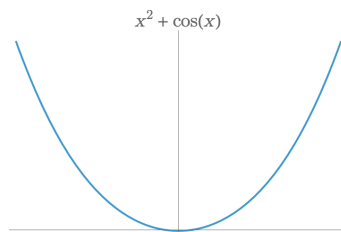
Statement

Illustration

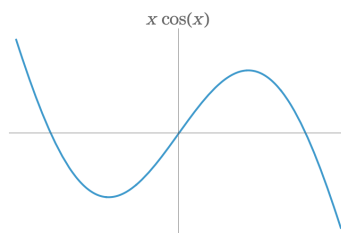
An even function times an even function is **even**



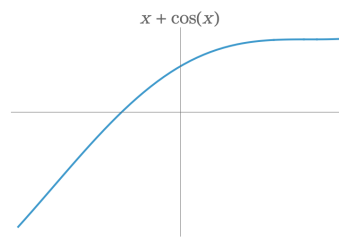
An even function plus an even function is **even**



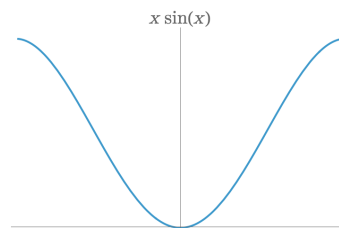
An even function times an odd function is **odd**



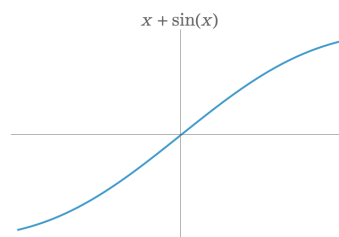
even function plus an odd function is **neither**



An odd function times an odd function is **even**



An odd function plus an odd function is **odd**



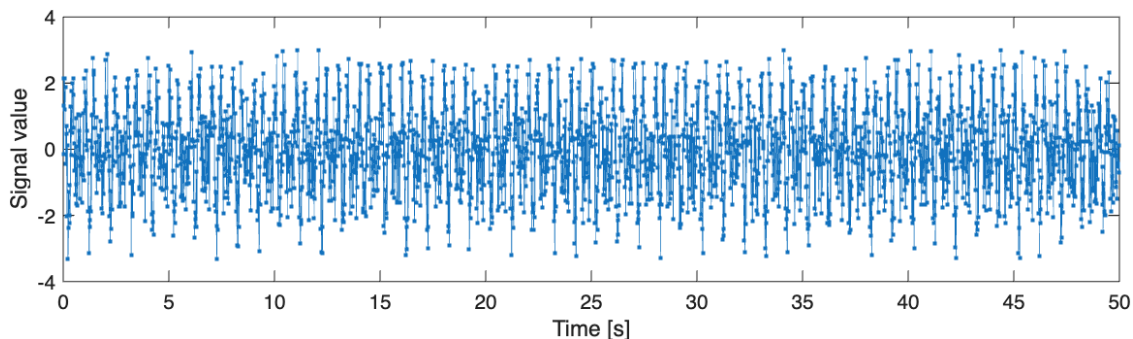
3 FFT in MATLAB

Download the following data file, which contains sampled measurements of some signal that was measured every 0.02 seconds. We suspect that the data was produced from some noisy signal containing only three discrete frequencies, which we will extract from this noisy data using MATLAB's `fft` function.

[Get data file](#)

[Get MATLAB file](#)

The data looks like this when plotted against time:



Identify a function of the kind

$$f_1(n_1t) + f_2(n_2t) + f_3(n_3t) + \text{noise} \quad (1)$$

that best represents this data, where n_1 , n_2 and n_3 are three integers that MATLAB will help you identify. f_1 , f_2 and f_3 are either sin or cos. The 'noise' term is not a real mathematical thing, but just a recognition that this data is not purely a sum of sines and cosines. Give your answer as a mathematical expression, and include a copy of the resulting MATLAB figure showing the correct expression in sub-plot 3. The first two sub-plots should not be changed.

i Note

Note that there is no multiplicate factor in Equation 1, i.e., each sin or cos function is multiplied by 1. In other words, you do not need to look for 'magnitudes' for each Fourier mode.

💡 Tip

If you don't know where to start, run `FFT_question.m` and look for peaks that are close to integers. Then, think about what function — made up of up to three sines or cosines — could best represent the given data.

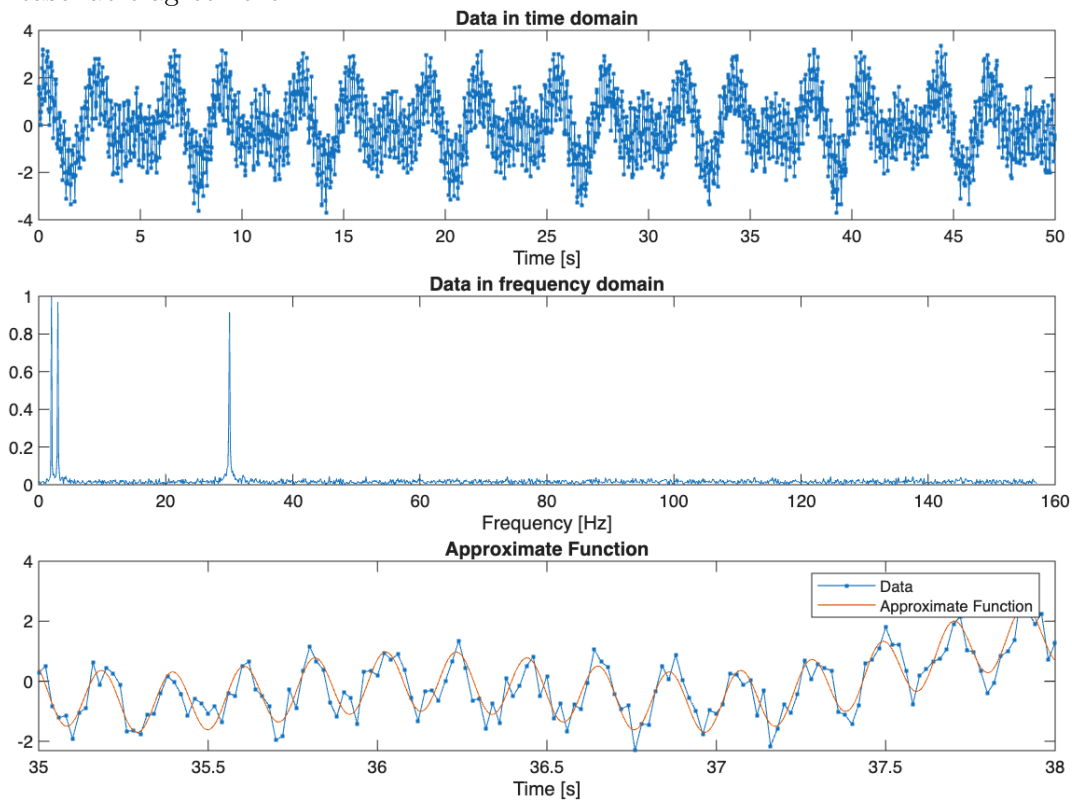
💡 Solution

After some trial and error, we see that the function

$$\sin(3t) + \cos(2t) + \cos(30t)$$

does a good job of representing the data.

We can see this below, where the approximate function is overlaid on top of the data, showing reasonable agreement.



4 Triangle Wave

Consider the triangle wave shown below. It is periodic with period 2π and is even about the vertical axis.

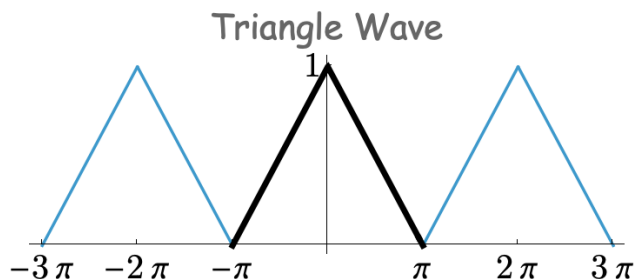


Figure 1: Triangle Wave

The triangle wave shown above can be defined using the following mathematical expression

$$f(t) = \begin{cases} 2 + \frac{t - \pi}{\pi} & -\pi < t < 0 \\ \frac{\pi - t}{\pi} & 0 < t < +\pi \end{cases},$$

$$f(t + 2\pi) = f(t)$$

Show by using piecewise integration of the Euler Formulas that

1. $a_n = \frac{4}{n^2\pi^2}$ for odd values of $n > 0$,
2. $a_n = 0$ for even values of n
3. $a_0 = 1/2$.
4. $b_n = 0$ for all values of n

and therefore, that the infinite Fourier series for this function is

$$f(t) = \frac{1}{2} + \frac{4}{\pi^2} \left(\cos t + \frac{\cos 3t}{9} + \frac{\cos 5t}{25} + \dots \right) \quad (2)$$

Solution

To solve this problem, we need to calculate a_n and b_n .

4.0.1 Expression for a_n

Let's start by computing the integral

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_{-\pi}^{+\pi} f(t) \cos nt \, dt. \\
 &= \frac{1}{\pi} \int_{-\pi}^0 \left(2 + \frac{t-\pi}{\pi} \right) \cos nt \, dt + \frac{1}{\pi} \int_0^{+\pi} \left(\frac{\pi-t}{\pi} \right) \cos nt \, dt \quad (3) \\
 a_n &= \frac{1}{\pi} \int_{-\pi}^0 2 \cos nt \, dt + \frac{1}{\pi} \int_{-\pi}^0 \left(\frac{t-\pi}{\pi} \right) \cos nt \, dt + \frac{1}{\pi} \int_0^{+\pi} \left(\frac{\pi-t}{\pi} \right) \cos nt \, dt
 \end{aligned}$$

and we shall compute the three terms above separately.

4.0.1.1 First term: $\int 2 \cos nt \, dt$ from $-\pi$ to 0

The first term in Equation 3 is

$$\frac{1}{\pi} \int_{-\pi}^0 2 \cos nt \, dt = \frac{2}{\pi} \left. \frac{\sin nt}{n} \right|_{-\pi}^0 = \frac{2}{\pi} \left(\frac{\sin 0}{n} - \frac{\sin(-n\pi)}{n} \right) = 0.$$

4.0.1.2 Second term: from $-\pi$ to 0

The second term in Equation 3 is

$$\begin{aligned}
 &\frac{1}{\pi} \int_{-\pi}^0 \left(\frac{t}{\pi} \cos nt - \frac{\pi}{\pi} \cos nt \right) dt \\
 &= \frac{1}{\pi} \left[\frac{t}{\pi n} \sin nt + \frac{1}{\pi n^2} \cos nt - \frac{1}{n} \sin nt \right]_{-\pi}^0
 \end{aligned}$$

where we have used integration by parts to find that

$$\int t \cos nt \, dt = \frac{t \sin nt}{n} + \frac{\cos nt}{n^2}. \quad (4)$$

Continuing with the second term in Equation 3, we evaluate each of the three terms at 0 and $-\pi$:

$$\begin{aligned}
 &= \frac{1}{\pi} \left[\frac{t}{\pi n} \sin 0 + \frac{1}{\pi n^2} \cos 0 - \frac{1}{n} \sin 0 \right] - \\
 &\frac{1}{\pi} \left[\left(\frac{-\pi}{\pi n} \sin(-n\pi) + \frac{1}{\pi n^2} \cos(-\pi n) - \frac{1}{n} \sin(-n\pi) \right) \right] \\
 &= \frac{1}{\pi} \left[0 + \frac{1}{\pi n^2} - 0 - 0 - \frac{1}{\pi n^2} \cos(-n\pi) - 0 \right] \\
 &= \frac{1}{\pi^2 n^2} - \frac{1}{\pi^2 n^2} \cos n\pi
 \end{aligned}$$

Now, we recall that for even n , $\cos n\pi = +1$ and for odd n , $\cos n\pi = -1$. This gives us that the first term in Equation 3 is:

$$\begin{cases} \frac{2}{n^2 \pi^2} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

4.0.1.3 Third term: from 0 to $+\pi$

We now turn our attention to the third term in Equation 3, which is

$$\begin{aligned} & \frac{1}{\pi} \int_0^{+\pi} \left(\frac{\pi}{\pi} \cos nt - \frac{t}{\pi} \cos nt \right) dt \\ &= \frac{1}{\pi} \left[\frac{1}{n} \sin nt - \left(\frac{t}{n\pi} \sin nt + \frac{1}{n^2\pi} \cos nt \right) \right]_0^{+\pi} \end{aligned}$$

where we have once again used integration by parts from Equation 4.

We now evaluate each of the three terms above at 0 and at $+\pi$, finding

$$\begin{aligned} &= \frac{1}{\pi} \left[\frac{\sin n\pi}{n} - \frac{\pi}{n\pi} \sin n\pi - \frac{\cos n\pi}{n^2\pi} - \left(\frac{1}{n} \sin 0 - \left(0 \sin 0 + \frac{\cos 0}{n^2\pi} \right) \right) \right] \\ &= \frac{1}{\pi} \left[0 - 0 - \frac{\cos n\pi}{n^2\pi} - 0 + 0 + \frac{1}{n^2\pi} \right] \\ &= \frac{1}{\pi^2 n^2} - \frac{1}{\pi^2 n^2} \cos n\pi \end{aligned}$$

which is identical to what we found earlier. So this, too is equal to

$$\begin{cases} \frac{2}{n^2\pi^2} & \text{if } n \text{ is odd, and} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

4.0.1.4 Overall expression for a_n

Thus, adding up the different terms, we find that

$$a_n = \begin{cases} \frac{4}{n^2\pi^2} & \text{if } n \text{ is odd, and} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

4.0.2 Expression for b_n

The next task is to calculate b_n . From the Euler formula, we know that

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{+\pi} f(t) \sin nt \, dt. \\ &= \frac{1}{\pi} \int_{-\pi}^0 \left(2 + \frac{t-\pi}{\pi} \right) \sin nt \, dt + \frac{1}{\pi} \int_0^{+\pi} \left(\frac{\pi-t}{\pi} \right) \sin nt \, dt \\ &= \frac{1}{\pi} \int_{-\pi}^0 2 \sin nt \, dt + \frac{1}{\pi} \int_{-\pi}^0 \left(\frac{t-\pi}{\pi} \right) \sin nt \, dt + \frac{1}{\pi} \int_0^{+\pi} \left(\frac{\pi-t}{\pi} \right) \sin nt \, dt \end{aligned} \tag{5}$$

4.0.2.1 First term: $\int 2 \sin nt \, dt$ from $-\pi$ to 0

The first term above is

$$\begin{aligned}
\frac{1}{\pi} \int_{-\pi}^0 2 \sin nt \, dt &= -\frac{2}{\pi} \frac{\cos nt}{n} \Big|_{-\pi}^0 = \frac{-2}{\pi} \left(\frac{\cos 0}{n} - \frac{\cos(-n\pi)}{n} \right) = 0. \\
&= \frac{-2}{n\pi} (1 - \cos n\pi) \text{ since } \cos \text{ is even.} \\
&= \begin{cases} 0 & n \text{ even} \\ \frac{-4}{n\pi} & n \text{ odd.} \end{cases}
\end{aligned}$$

Where we have deduced the above from the fact that $\cos n\pi = 1$ for even n and $\cos n\pi = -1$ for odd n .

4.0.2.2 Second term: from $-\pi$ to 0

The second term in Equation 5 will once again require integration by parts. We have

$$\begin{aligned}
&\frac{1}{\pi} \int_{-\pi}^0 \left(\frac{t-\pi}{\pi} \right) \sin nt \, dt \\
&= \frac{1}{\pi} \left[\frac{-t \cos nt}{n\pi} + \frac{\sin nt}{n^2\pi} + \frac{\cos nt}{n} \right]_{-\pi}^0
\end{aligned}$$

where we have used integration by parts to find that

$$\int t \sin nt \, dt = -\frac{t \cos nt}{n} + \frac{\sin nt}{n^2} \tag{6}$$

Continuing with the second term in Equation 5, we evaluate each of the three terms at 0 and $-\pi$:

$$\begin{aligned}
&\frac{1}{\pi} \left[\frac{-0 \cos 0}{n\pi} + \frac{\sin 0}{n^2\pi} + \frac{\cos 0}{n} \right] - \\
&\frac{1}{\pi} \left[\frac{-(-\pi) \cos(-n\pi)}{n\pi} + \frac{\sin(-n\pi)}{n^2\pi} + \frac{\cos(-n\pi)}{n} \right] \\
&= \frac{1}{\pi} \left[0 + 0 + \frac{1}{n} \right] - \frac{1}{\pi} \left[\frac{\cos n\pi}{n} + 0 + \frac{\cos n\pi}{n} \right] \\
&= \frac{1}{\pi} \left(\frac{1}{n} - \frac{2 \cos n\pi}{n} \right) \\
&= \begin{cases} \frac{1}{\pi} \left(\frac{1}{n} - \frac{2}{n} \right) = -\frac{1}{n\pi} & n \text{ even} \\ \frac{1}{\pi} \left(\frac{1}{n} - \frac{-2}{n} \right) = \frac{3}{n\pi} & n \text{ odd} \end{cases}
\end{aligned}$$

4.0.2.3 Third term: from 0 to $+\pi$

We now turn our attention to the third term in Equation 5, which is

$$\begin{aligned}
&\frac{1}{\pi} \int_0^{+\pi} \left(\frac{\pi-t}{\pi} \right) \sin nt \, dt \\
&= \frac{1}{\pi} \left[\frac{-\cos nt}{n} + \frac{t \cos nt}{n\pi} - \frac{\sin nt}{n^2\pi} \right]_0^{+\pi}
\end{aligned}$$

where we have once again used integration by parts from Equation 6.

We now evaluate each of the three terms above at 0 and at $+\pi$, finding

$$\begin{aligned} &= \frac{1}{\pi} \left[\frac{-\cos n\pi}{n} + \frac{\pi \cos n\pi}{n\pi} - \frac{\sin n\pi}{n^2\pi} - \left(\frac{-\cos 0}{n} + \frac{0 \cos 0}{n\pi} - \frac{\sin 0}{n^2\pi} \right) \right] \\ &= \frac{1}{\pi} \left[\frac{-\cos n\pi}{n} + \frac{\cos n\pi}{n} - 0 + \frac{1}{n} + 0 + 0 \right] \\ &= \frac{1}{n\pi} \end{aligned}$$

4.0.2.4 Overall expression for b_n

We have a lot of terms to collect now. Let's write out what b_n is in the even and odd cases.

- Even n :

$$b_n = 0 - \frac{1}{n\pi} + \frac{1}{n\pi}$$

which amounts to zero.

- Odd n :

$$b_n = \frac{-4}{n\pi} + \frac{3}{n\pi} + \frac{1}{n\pi}$$

which also amounts to zero.

Thus, we have successfully shown that $b_n = 0$ for all n , QED.

4.0.3 Expression for a_0

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(t) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^0 \left(2 + \frac{t-\pi}{\pi} \right) dt + \frac{1}{2\pi} \int_0^{+\pi} \left(\frac{\pi-t}{\pi} \right) dt \end{aligned}$$

At this point, we note that Figure 1 depicts an odd function and the area under the graph should be the same to the left and to the right. So for simplicity I will just find the area under the graph for the right-hand side, since that integral looks simpler. Then I can multiply by 2.

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{+\pi} \left(\frac{\pi-t}{\pi} \right) dt \\ &= \frac{1}{2\pi^2} \int_0^{+\pi} \pi dt - \frac{1}{\pi^2} \int_0^{+\pi} t dt \\ &= \frac{1}{2\pi^2} \left. t\pi \right|_0^{+\pi} - \frac{1}{2\pi^2} \left. \frac{t^2}{2} \right|_0^{+\pi} \\ &= \frac{\pi^2}{2\pi^2} - \frac{\pi^2}{4\pi^2} = \frac{1}{4} \end{aligned}$$

and therefore, multiplying the above result by 2 to account for the area under the left-half of the graph of $f(t)$, we have

$$a_0 = 2 \times \frac{1}{4} = \frac{1}{2}.$$

We have therefore successfully shown that the Fourier series for Equation 2 is

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

where $a_n = \begin{cases} \frac{4}{n^2\pi^2} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$

and $b_n = 0$ for all n .