Problem Set 7

ENGR 12, Spring 2026.

Due Date	Tue, Mar 17
Turn in link	Gradescope
URL	`emadmasroor.github.io/E12-S26/Homework/HW7`

Points Distribution

Please note that each of the following grade items is a **single rubric item**. Each rubric item is scored on a four-level scale of 3-2-1-0. You may wish to take this into account when deciding how to allocate your efforts to each problem.

Problem	Part	% Weightage
Problem 1		20
Problem 2	2.1	20
	2.2	20
Problem 3		20
Problem 4		20

1 Classifying second-order systems

1.1 Under-, over- and critically- damped systems

For each of the following systems, determine if you expect the free response to be over-damped, under-damped, or critically damped. Show your work. If a system does not seem to fall into any category, sketch the shape you expect its solution $x(t)$ to take.

1. $$\ddot{x} + 4\dot{x} + 3x = 0$$
2. $$\ddot{x} + 3\dot{x} + 4x = 0$$
3. $$2\ddot{x} + \dot{x} + 3x = 0$$
4. $$3\ddot{x} + 6\dot{x} + 3x = 0$$
5. $$\ddot{x} + 4\dot{x} - 3x = 0$$
6. $$2\ddot{x} - 4\dot{x} + 3x = 0$$

> **Tip**
>
> Your sketch does not need to be accurate. Notice that no initial conditions are provided, so we are not looking for a correct graph anyway. Just think about what the solution should qualtiatively look like.

2 Forced (Step) Response of second-order systems

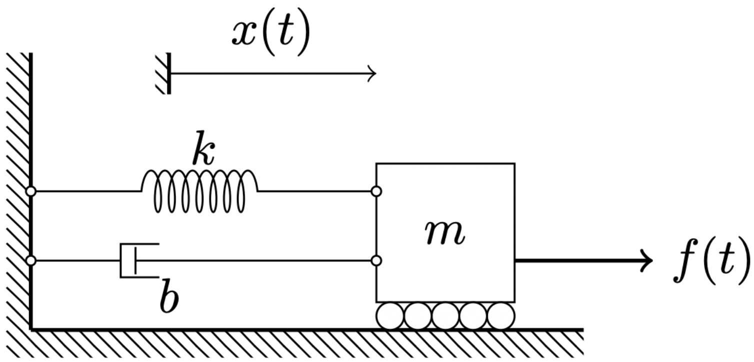


Figure 1: Spring-mass-damper system. Assume that x has been chosen such that when $x = 0$, there is no push or pull in the spring.

> Tip
>
> When a question asks for 'the unit impulse response', the 'unit step response', or really the 'something response', what it's really asking is this:
> Set up the governing differential equation for the system, with the input on the right-hand side. In E12, this will usually look like $\ddot{x} + \dot{x} + x = f(t)$. Let the input — here, $f(t)$ — be equal to the 'something' the response to which you are asked to find. Then, what is $x(t)$?
> Note that we usually do not integrate the equation to find the answer, but use other techniques.

2.1 Deriving an expression for the step response

> Warning
>
> - This question has been clarified to indicate that the spring-mass-damper is underdamped.
> - A missing factor of m in Equation 1 was added on Friday March 13

Show that the forced response of an **underdamped** spring-mass-damper system, such as the one shown above, to a 'step function force' equal to c, i.e., the solution of the differential equation

$$m\ddot{x} + b\dot{x} + kx = cu_s(t),$$

is

$$x(t) = \frac{c}{m(r^2 + \omega^2)} \left[1 - e^{rt} \left(\cos \omega t - \frac{r}{\omega} \sin \omega t\right)\right]. \tag{1}$$

Here, r is the real part of the roots of the characteristic polynomial, and ω the imaginary part of the roots of the characteristic polynomial. You should not explicitly leave m, b and k in your answer.

> ⚠ Warning
>
> The hand-written part of lecture 14 relevant to this problem had some mistakes; it has been removed from the lecture slides for this reason.

In the process of showing the above, **also write down the frequency-domain expression** for the forced response, i.e., write down an expression for $X(s)$ in terms of ω, r and c.

> Tip
>
> It is possible to solve this problem by using a partial fractions expansion such as
>
> $$\frac{1}{s}\frac{1}{ms^2+bs+k} = \frac{P}{s} + \frac{Q}{s-s_1} + \frac{R}{s-s_2}.$$
>
> However, the above strategy is **not** recommended because it requires using complex Q and R, which is a huge pain. Instead, it is recommended that you use the following partial fractions expansion:
>
> $$\frac{1}{s}\frac{1}{ms^2+bs+k} = \frac{P}{s} + Q\frac{s-r}{(s-r)^2+\omega^2} + R\frac{\omega}{(s-r)^2+\omega^2}, \tag{2}$$
>
> where P, Q and R are real constants to be determined (by you).

2.2 Using specific numbers.

For this part, it is OK to just use Equation 1 directly without having to solve any equations.

2.2.1 Expression for $x(t)$ with numbers

For $m = 1, b = 2, k = 4$ and $c = 2$, write down the expression for $x(t)$ using numbers and functions. There should be no free parameters here.

2.2.2 Predicting the steady-state response

Recall that the 'steady-state response' to an input is how a system behaves in the long run due to the input it receives. For $m = 1, b = 2, k = 4$ and $c = 2$, calculate — without making a plot — the steady-state response, i.e. what value(s) $x(t)$ takes in the long run.

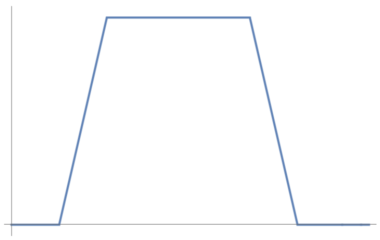
2.2.3 Making a plot

1. For $m = 1, b = 2, k = 4$ and $c = 2$, make a plot of $x(t)$ for the first ten seconds.
2. **Either** using your plot **or** using math, determine the time at which the spring is the farthest from its original position, and write the value of t and of x for this instant.

3 The 'top-hat response'

Once again, we would like to examine our system's behavior in response to a force that is switched on for a period of time and then switched off. We will use a 'top hat' function like before, which consists of a ramp up, a constant period, and then a ramp back down to zero.

A spring-mass-damper system is subjected to a force given by the graph shown below.



This function can be specified in MATLAB using

```
function output = f1(t)
    % Builds a 'hat' from t = 2 to t = 12
    % using ramp functions.
    output = + heaviside(t-2).*5.*(t-2) + ...
             - heaviside(t-4).*5.*(t-4) + ...
             - heaviside(t-10).*5.*(t-10) + ...
             + heaviside(t-12).*5.*(t-12);
end
```

The differential equation can be implemented in MATLAB using

```
function dydt = rhs1(t,y,m)
% Define the right-hand side function of differential equation. It takes as
% input the value of m, which is a parameter we will change, in addition to
% the usual "t" and "y", where "y" is the state variable.
    b = 2; k = 4;

    x = y(1);
    xdot = y(2);

    dy1dt = xdot;
    dy2dt = (-b*xdot - k*x + f1(t))/m;

    dydt= [dy1dt;dy2dt];
end
```

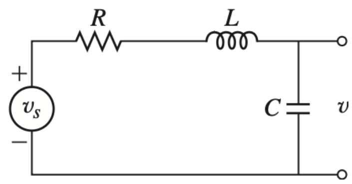
Use the following starter code to investigate **three values of** m that show qualitatively different kinds of behaviorss. Describe how the behaviors you see are qualitatively different from each other.

```
% Define a time domain with lots of values
N     = 1000;
tvals = linspace(0,40,N);

% Call ode45 with initial conditions of zero using various values for m.
[t1,x1] = ode45(@(t,x) rhs1(t,x,1), tvals, [0;0]);

% Plot response
clf;
yyaxis right;
plot(tvals,f1(tvals),'--',"LineWidth",1,'Color',"black");  hold on;
ylabel("Applied force f(t)")
yyaxis left;
plot(tvals,x1(:,1),'-',"LineWidth",2,"Color","blue"); hold on;
ylabel("Response x(t)")

% Make it pretty
set(gca,"FontSize",14);
pbaspect([3,1,1]);
xlabel("Time");
```

What to turn in

Do **not** turn in `.m` files. It is sufficient to provide only your MATLAB figure and an accompanying explanation, which should be a few sentences long (up to one paragraph).

4 RLC Circuits



For the RLC circuit shown, it is known that the governing equation for the voltage across the capacitor is

$$LC\ddot{v} + RC\dot{v} + v = v_s.$$

When there is no input voltage and just a free response, what is the

1. Undamped natural frequency,
2. Damping ratio, and
3. Damped natural frequency

for the system? Give your answers in terms of R, L and C.