Problem Set 8 Solutions

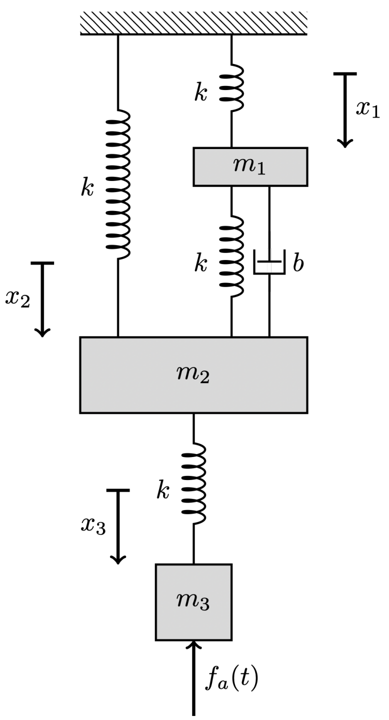
ENGR 12, Spring 2026.

Solutions

1 Deriving the equations

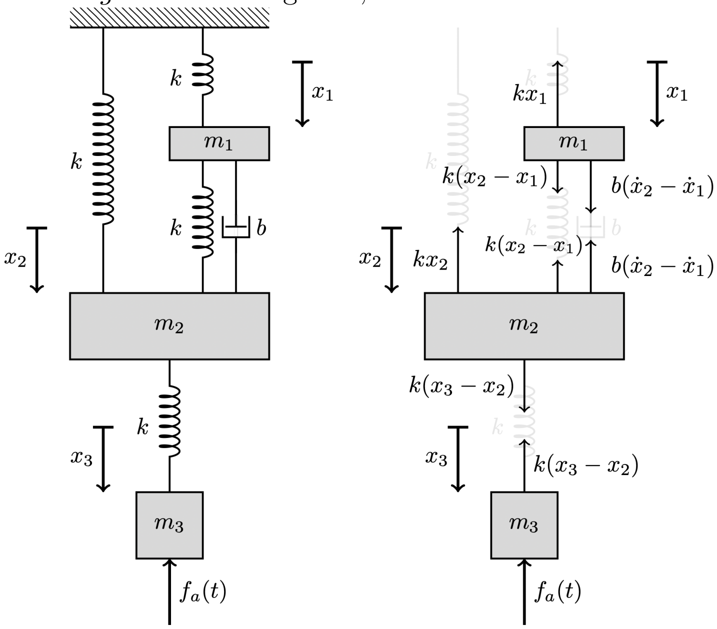
Draw correct free-body diagrams and find the governing differential equation(s) for the following system. There should be three second-order differential equations for x_1, x_2 and x_3. Assume that all three lengths are defined such that, when $x_1 = x_2 = x_3 = 0$, there is no restoring force in any of the springs.

> **Warning**
>
> Don't forget the gravity term!



Solution

Let us present below the free-body diagrams for each mass. For neatness, we will omit the term mg from our diagrams, but we will remember to include it in the equations.



1. FBD of mass 1:
$$m_1\ddot{x}_1 = k(x_2 - x_1) + b(\dot{x}_2 - \dot{x}_1) - kx_1 + m_1 g$$
2. FBD of mass 2:
$$m_2\ddot{x}_2 = k(x_3 - x_2) - kx_2 - k(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1) + m_2 g$$
3. FBD of mass 3:
$$m_3\ddot{x} = -f_a(t) + m_3 g - k(x_3 - x_2)$$

2 Two Masses on a moving cart

Consider the system shown below, in which two masses M_1 and M_2 are placed on a moving cart with mass M_3. There is friction between the masses and the cart, following the usual 'element law' of friction being proportional to the relative velocity between the two surfaces. A force is applied to mass M_1 to the right with magnitude $f_a(t)$.

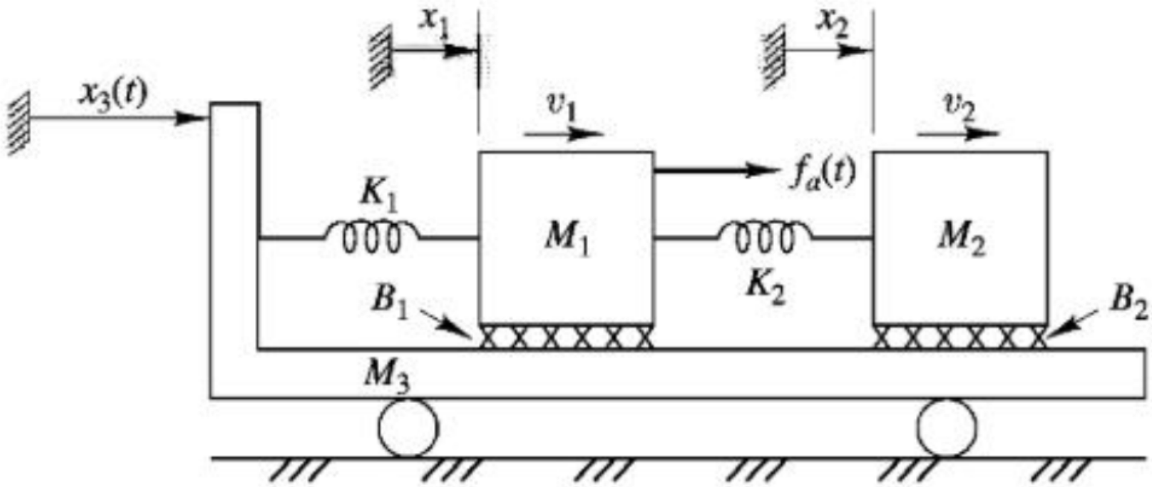


Figure 1: Two masses are connected to each other and to a cart with springs.

2.1 State Variables

Determine the number of state variables in this system and write down mathematical symbols for each of them.

Solution

There are five energy-storing elements in this system: the three masses and the two springs. Thus, there appear to be five state variables. They are x_1, x_2, $\dot{x}_3$, $\dot{x}_1$ and $\dot{x}_2$.

2.2 Input Force

For this part, assume that the cart is forced to be at rest at all times and $x_3(t) = \dot{x}_3(t) = \ddot{x}_3(t) = 0$. The system is put into motion through a force $f_a(t)$. Ignore initial conditions.

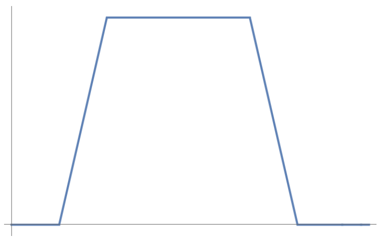
1. Draw all relevant free-body diagrams for this system. Your diagrams should have the correct arrows, labels, and signs.
2. Write down the governing differential equations for this system.
3. Using the state variables that you identified in Section 2.1 or a subset of them, write down the governing differential equations for this system in the form $\dot{x} = A\,x + b$, where x is a vector of state variables, A is the system matrix, and b contains information about the applied force.
4. Use MATLAB to solve this system of equations numerically for an applied force $f_a(t)$ that is given by the 'top-hat function' from HW 7. Generate a plot of the resulting solution and **turn in a copy of the modified file** `prob12_rhs.m` (in the form of a screenshot or text, not a `.m` file) **as well as an image** of the plot. Note that the system should start from rest. You may make use of the skeleton code provided below. The only significant change that you need to make is in the file named `prob12_rhs.m`. The parameter values are:

- $m_1 = m_2 = 1$

- $k_1 = k_2 = 2$
- $b_1 = b_2 = 0.5$

5. Repeat the MATLAB solution with a different value for $k_1 = 10$, keeping all other parameters the same. **Turn in the resulting plot**.

The input as a function of time looks like the following figure:



Solution

1. TBD
2. The governing differential equations are (note that $x_3 = 0$)

$$\begin{aligned} M_1\ddot{x}_1 &= f_a(t) + K_2(x_2 - x_1) - K_1x_1 - B_1\dot{x}_1 \\ M_2\ddot{x}_2 &= \quad - K_2(x_2 - x_1) \quad - B_2\dot{x}_2 \end{aligned}$$

3. Using four state variables x_1, $\dot{x}_1$, x_2 and $\dot{x}_2$, we can write these equations as

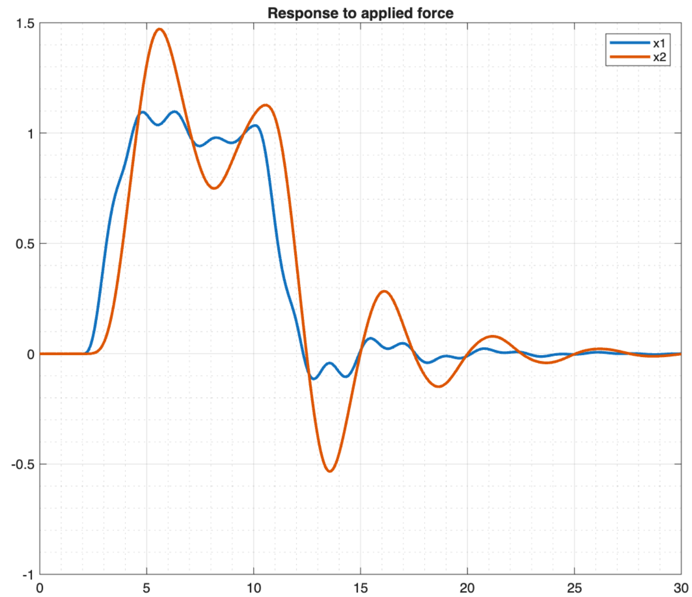
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{(K_1+K_2)}{M_1} & -\dfrac{B_1}{M_1} & \dfrac{K_2}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{K_2}{M_2} & 0 & -\dfrac{K_2}{M_2} & -\dfrac{B_2}{M_2} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f_a(t)/M_1 \\ 0 \\ 0 \end{bmatrix}$$

4. The missing code is shown below in a complete copy of the `prob12_rhs.m` file.

Listing 1 `prob12_rhs.m`

```
function dydt = prob12_rhs(t,y,params)
% r.h.s. of ode for problem 1.2 with applied force.

k1 = params.k1;
k2 = params.k2;
m1 = params.m1;
m2 = params.m2;
b1 = params.b1;
b2 = params.b2;

fa = @(t)    + heaviside(t-2).*5.*(t-2) + ...
             - heaviside(t-4).*5.*(t-4) + ...
             - heaviside(t-10).*5.*(t-10) + ...
             + heaviside(t-12).*5.*(t-12);

x1 = y(1);
x1dot = y(2);
x2 = y(3);
x2dot = y(4);

x1ddot = fa(t)/m1 + (k2/m1)*(x2-x1) - (k1/m1)*x1 - (b1/m1)*(x1dot);
x2ddot = -(k2/m2)*(x2-x1) - (b2/m2)*(x2dot);

dydt = [x1dot;x1ddot;x2dot;x2ddot];
end
```

5. The plot is shown below.



2.3 Input displacement

For this part, we no longer apply a force to M_1 and therefore set $f_a(t) = 0$. Motion is imparted to the system by imposing a kinematic motion $x_3(t)$ to the cart. Here, $x_3(t)$ is some known function of time, whose derivatives can also be calculated and therefore are also known functions of time.

1. Re-write the governing equations for this system under the given conditions.
2. Write the new governing equations in the form $\dot{y} = Ay + b$, where y is a vector containing $[x_1, \dot{x}_1, x_2, \dot{x}_2]$ and b is a vector containing the 'inputs'.
3. Write down an expression for the force that would have to be applied to M_3 (in the right-ward direction) to bring about the same motion as that brought about by the given displacement $x_3(t)$. Note that this expression will contain some terms that are already known, and some terms that, while not explicitly known, have a corresponding differential equation available.

Solution

1. The governing equations should now take into account the relative velocity between the cart and the masses. The equations are

$$\begin{aligned} M_1\ddot{x}_1 &= K_2(x_2 - x_1) - K_1(x_1 - x_3) - B_1(\dot{x}_1 - \dot{x}_3) \\ M_2\ddot{x}_2 &= -K_2(x_2 - x_1) \qquad\qquad - B_2(\dot{x}_2 - \dot{x}_3) \end{aligned}$$

2. In the required form, the equations are

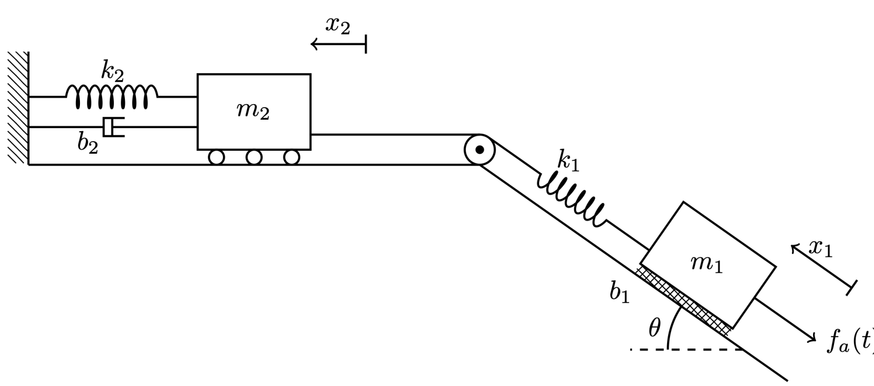
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{(K_1+K_2)}{M_1} & -\dfrac{B_1}{M_1} & \dfrac{K_2}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{K_2}{M_2} & 0 & -\dfrac{K_2}{M_2} & -\dfrac{B_2}{M_2} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{B_1}{M_1}\dot{x}_3 + \dfrac{K_1}{M_1}x_3 \\ 0 \\ \dfrac{B_2}{M_2}\dot{x}_3 \end{bmatrix}$$

3. The force that would need to be applied is given by

$$f = -M_3\ddot{x}_3 - B_2(\dot{x}_3 - \dot{x}_2) - B_1(\dot{x}_3 - \dot{x}_1) + K_1(x_1 - x_3).$$

3 Mass hanging on inclined plane

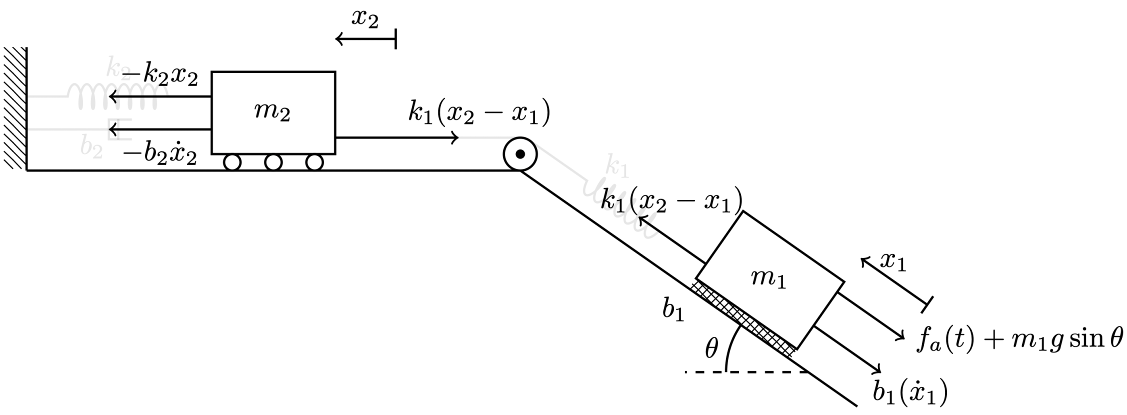
Two masses are connected to each other with an ideal pulley. One of them is on a friction-full inclined plane while the other rests on friction-less wheels. The coordinates x_1 and x_2 are measured relative to some fixed locations, which have been chosen such that when $x_1 = x_2 = 0$, there is no tension or compression in the springs. A force $f_a(t)$ is applied to the inclined mass as shown.



3.1 Free body diagrams

Draw the free body diagrams for this system.

Solution



3.2 Governing Equations

Write down two second-order differential equations for the motion of this system.

Solution

$$m_2\ddot{x}_2 = -k_2x_2 - b_2\dot{x}_2 - k_1(x_2 - x_1)$$
$$m_1\ddot{x}_1 = \quad - b_1\dot{x}_1 + k_1(x_2 - x_1) - m_1 g \sin\theta - f_a(t)$$

3.3 Equilibrium value

Under gravity alone (i.e., $f_a(t) = 0$), this system won't naturally stay at $x_1 = x_2 = 0$. Instead, it will settle to some equilibrium value. Determine mathematical expressions for the equilibrium value of x_1 and of x_2, in terms of the parameters given in the problem and g, the acceleration due to gravity..

Solution

The equilibrium value arises from the governing equations in the condition when the system has become stationary.

$$m_2\cancel{\ddot{x}_2} = -k_2x_2 - b_2\cancel{\dot{x}_2} - k_1(x_2 - x_1)$$
$$m_1\cancel{\ddot{x}_1} = \quad - b_1\cancel{\dot{x}_1} + k_1(x_2 - x_1) - m_1 g \sin\theta - \cancel{f_a(t)}$$

This gives us two equations for x_1 and x_2.

$$0 = -k_2x_2 - k_1(x_2 - x_1)$$
$$0 = k_1(x_2 - x_1) - m_1 g \sin\theta$$

Adding up these equations, we find that

$$-k_2x_2 - m_1 g \sin\theta = 0 \implies x_2 = \boxed{-\frac{m_1 g}{k_2}\sin\theta}$$

and thus,

$$x_1 = \frac{k_2 + k_1}{k_1} x_2 = \boxed{-\frac{k_2 + k_1}{k_1 k_2} m_1 g \sin\theta}$$

The boxed quantities are the equilibrium values of x_2 and x_1.

Hint

If the system is initialized with $x_1 = x_2 = \dot{x}_1 = \dot{x}_2 = f_a(t) = 0$, the values of x_1 and x_2 follow the graph shown below. The property values used are shown in the table below.

Table 1: Ignore Units

Parameter	Quantity
m_1	1
m_2	3
b_1	2
b_2	1
k_1	5
k_2	10
θ	30°

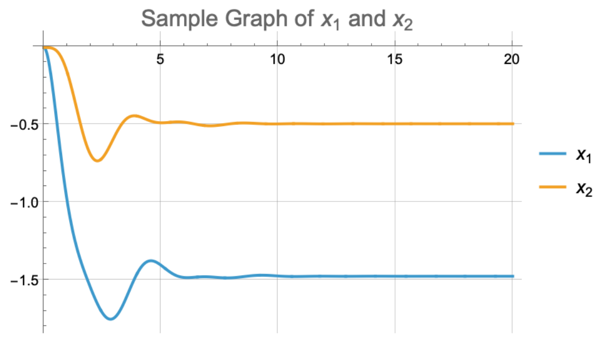


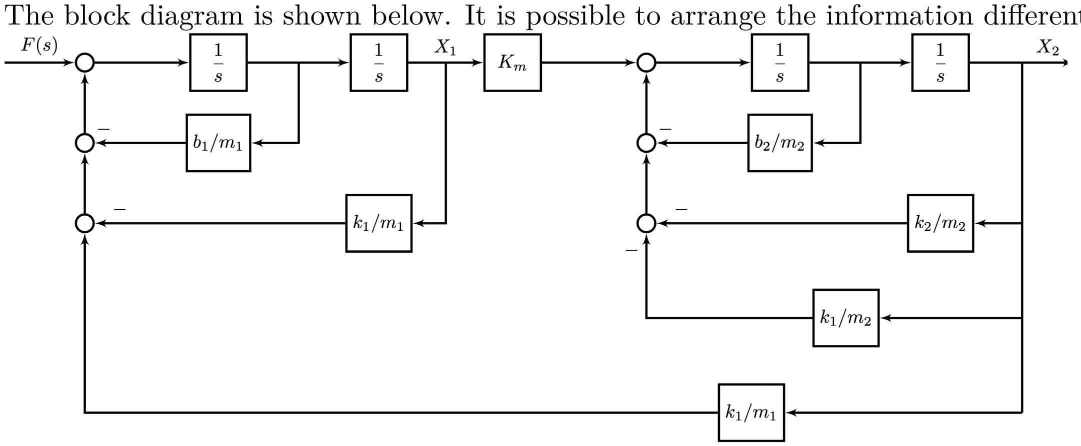
Figure 2

3.4 Block Diagram

Make a block diagram for this system. In your block diagram, gather the terms related to gravity and the terms related to the applied force $f_a(t)$ in a single arrow, which you should show as an input to your system.

Solution

The block diagram is shown below. It is possible to arrange the information differently.



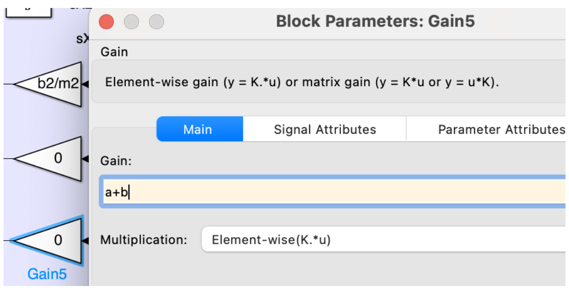
3.5 Simulink

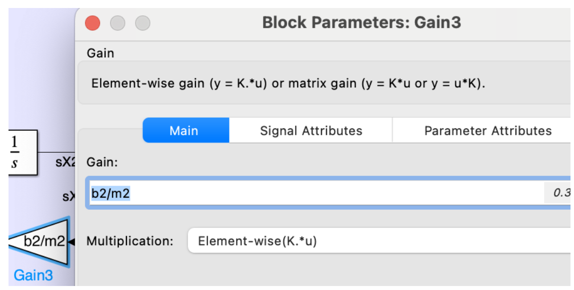
Get the Simulink file `model1.slx`, which contains an incomplete simulation of the hanging mass system. Please also place the accompanying files `inclinedhang_runsimulink.m` and `inclinedhang_setupvariables.m` in the same folder as your Simulink model file.

Get zip file

3.5.1 Populate all gain blocks

Most rectangular blocks in a traditional block diagram are indicated in Simulink by triangular 'gain' blocks. Determine what goes in each 'gain' block. For example, if the sum of two MATLAB variables `a+b`, enter it like so. Some blocks have been correctly populated for you.

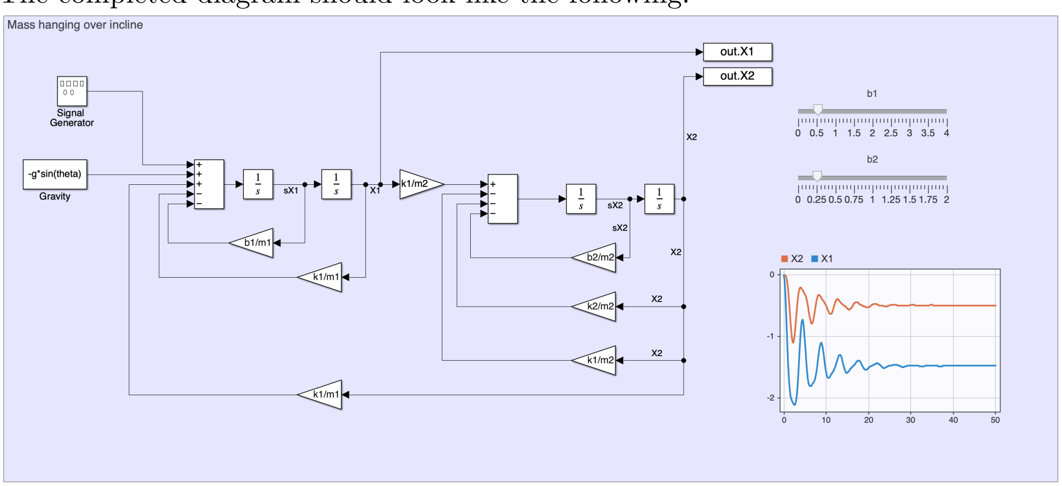




Turn in a screenshot of your completed Simulink diagram.

Solution

The completed diagram should look like the following.



3.5.2 Overdamped?

Does the system as a whole seem overdamped, underdamped, or critically damped when you use the highest allowed values of $b_1 = 4$ and $b_2 = 2$? It is sufficient to write a single-word answer to this question.

Solution

This system seems underdamped, since there is some oscillation before it settles into a steady-state value, albeit small.

3.5.3 Explore Parameters

While setting the applied force $f_a(t)$ to zero, set the sliders to different values for b_1 and b_2 between the ranges provided. Select some values for which you see qualitatively different behavior, and save the corresponding plots. There are many ways to do this; you can use `inclinedhang_runsimulink.m` or you can write your own script. If you're really having trouble, a screenshot of the "view" panel in Simulink will also suffice.

1. Make one set of plots using the values from Table 1. Your graph should look very similar to Figure 2.
2. Choose some values of b_1 and b_2 that show more "bouncy-ness" and look qualitatively different from Figure 2.
3. Choose some values of b_1 and b_2 that show much less "bouncy-ness".

For each of the above cases, a plot together with values of b_1 and b_2 (either in the form of a legend/label or as accompanying text) is all you need to turn in. There will be three plots in total, each of them showing both x_1 and x_2.

Solution

The following figures were produced using a numerical solution of the governing equations in Mathematica.