

# Lecture 25

## E12 Linear Physical Systems Analysis

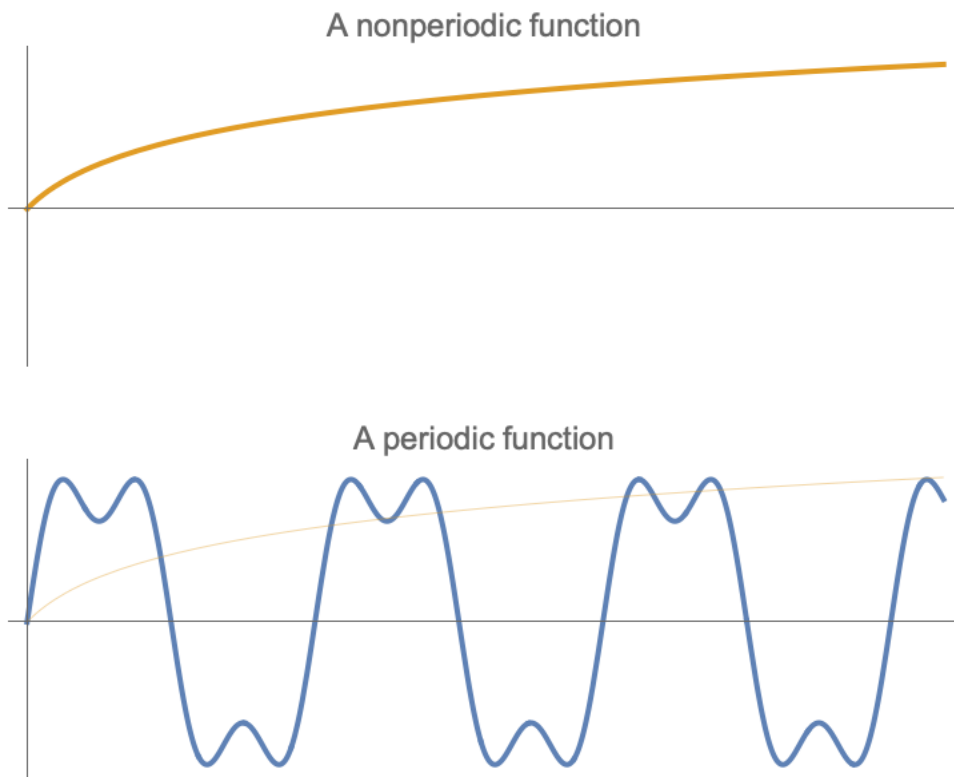
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### Periodic Functions

Some mathematical functions have the remarkable property of **repeating** themselves.



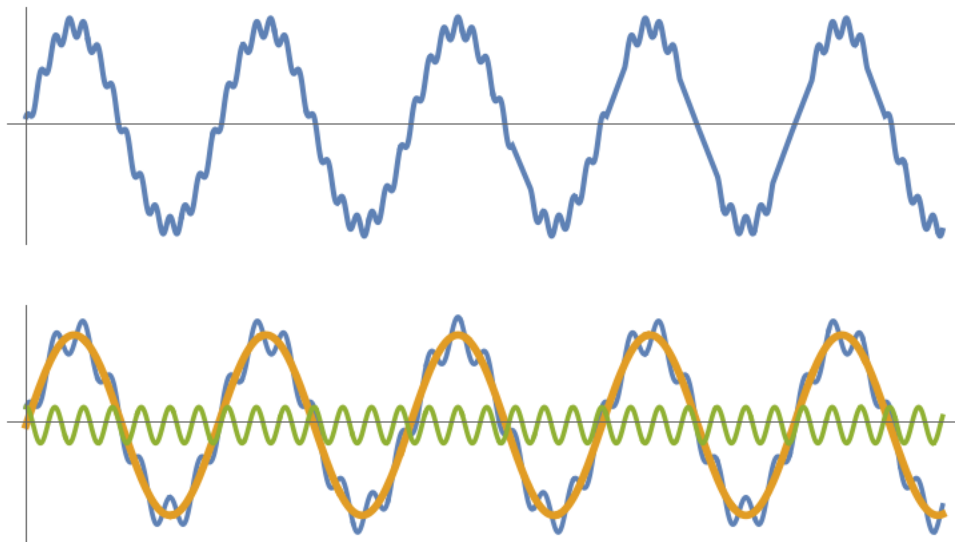
### Defining what a periodic function is

The formal definition of a periodic function is:

- A function  $f(t)$  is said to be **periodic** in  $t$  if, for some  $T$ ,  $f(t + nT) = f(t)$  for all integer  $n$ .
- If this is so, the number  $T$  is known as the period of the function  $f$ .
- We can describe a periodic function using one of three terms:

- Its period  $T$
- Its frequency  $f = 1/T$
- Its angular frequency  $\omega = 2\pi f = 2\pi/T$
- As an example,  $g(t) = \sin 3t$  is a periodic function because we can find a value of  $T$  that will make the equation  $g(t) = g(t + nT)$  true for all  $T$ .
  - Task: what is this special value of  $T$ ?
- As another example,  $g(t) = \log(t + 1)$  is a nonperiodic function because we cannot find any value of  $T$  that will make the equation  $g(t) = g(t + nT)$  true for all  $T$ .

### Periodic functions made up of multiple sines and cosines.

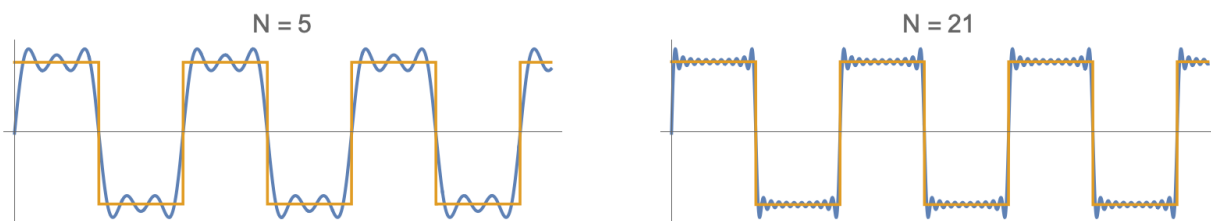


- Sometimes, a function will ‘contain’ two very different frequencies.
- Often, very high frequencies contribute to the “noise” in a single.
- In this case, we have two functions:

$$\sin 3t + 0.2 \cos 20t$$

### The Fourier theorem (illustration)

**Any** periodic function can be represented as the sum of sines and cosines.



### Mathematical Statement of the Fourier Theorem

A periodic function  $x(t)$  with period  $P$  is equal to the convergent infinite series

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{2\pi nt}{P}\right) + b_n \sin\left(\frac{2\pi nt}{P}\right) \right)$$

$$x(t) \approx a_0 + \sum_{n=1}^N \left( a_n \cos\left(\frac{2\pi nt}{P}\right) + b_n \sin\left(\frac{2\pi nt}{P}\right) \right)$$

for  $a_n, b_n$  given by the **Euler Formulas**:

$$a_0 = \frac{1}{P} \int_{-P/2}^{+P/2} x(t) dt$$

$$a_n = \frac{2}{P} \int_{-P/2}^{+P/2} x(t) \cos\left(\frac{2\pi nt}{P}\right) dt \quad n > 0$$

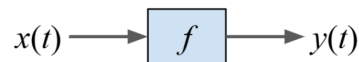
$$b_n = \frac{2}{P} \int_{-P/2}^{+P/2} x(t) \sin\left(\frac{2\pi nt}{P}\right) dt \quad n > 0$$

**Why do we care**

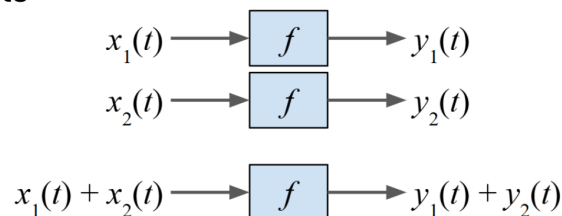
**That any periodic function can be written as the sum of sines and cosines?**

## Superposition in linear systems

Linear systems transform a time-varying input into a time-varying output



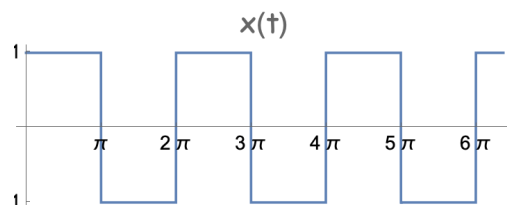
The response to the sum of inputs is the sum of the responses to individual inputs



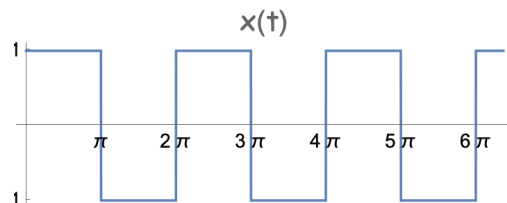
**Computing the Fourier Series for small  $N$** 

$$x(t) \approx a_0 + \sum_{n=1}^N \left( a_n \cos \left( \frac{2\pi n t}{P} \right) + b_n \sin \left( \frac{2\pi n t}{P} \right) \right)$$

- If a truly infinite number of terms are added up, i.e., if  $N = \infty$  we have an exact equality between the left and right hand sides.
- What happens when  $N$  is finite? Let's examine the case when  $N = 5$
- Let  $x(t)$  be given by the following function

**Computing the coefficients**

$$x(t) \approx a_0 + \sum_{n=1}^N \left( a_n \cos \left( \frac{2\pi n t}{P} \right) + b_n \sin \left( \frac{2\pi n t}{P} \right) \right)$$

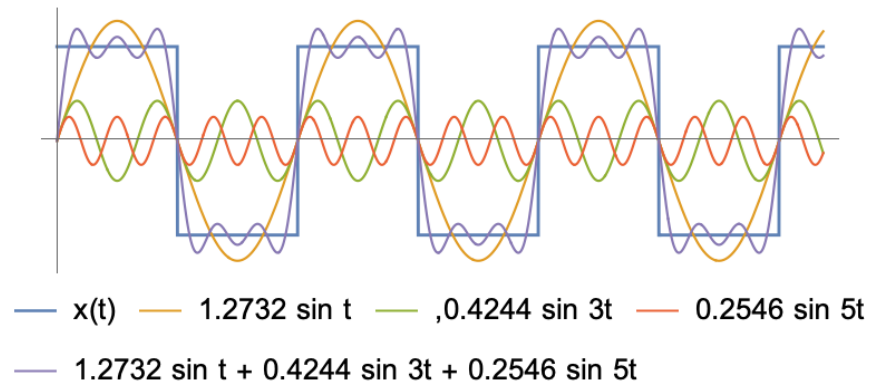


First, we compute the terms  $a_n$  and  $b_n$  for  $N = 5$ . Using the Euler Formulas, we find the following values

Term	Value	Term	Value
$a_0$	0		
$a_1$	0	$b_1$	1.2732
$a_2$	0	$b_2$	0
$a_3$	0	$b_3$	0.4244
$a_4$	0	$b_4$	0
$a_5$	0	$b_5$	0.2546

The Fourier Theorem tells us that

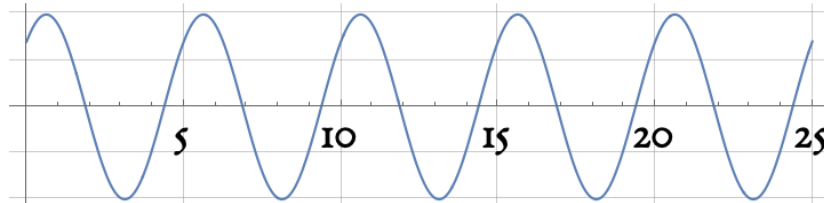
$$x(t) \approx 1.2732 \sin t + 0.4244 \sin 3t + 0.2546 \sin 5t$$

**Visualizing the terms**

- Each individual term does **not** resemble  $x(t)$
- But their cumulative sum does.
- The terms are simply sine functions with different frequencies.
- The magnitude of each term's contribution — i.e., the amplitude of each constituent sinusoid — is given by the Euler formulas.

**Harmonics and the Fundamental Frequency**

- A function with a period of 5 seconds repeats itself every 5 seconds.
- It **also repeats itself** every  $2 \times 5$  seconds, every  $3 \times 5$  seconds, every  $4 \times 5$  seconds, ...
- What is the period of this function?



- If a function  $f(t)$  is periodic, we refer to its **smallest** period as its **fundamental period**
- The frequency associated with the fundamental period, i.e.,  $f = 1/T$ , is known as the **fundamental frequency**
- If a function  $x(t)$  has fundamental frequency  $f_0$ , then its second harmonic is the frequency  $2f_0$ , its third harmonic is the frequency  $3f_0$ , and so on.
- Connection to musical notes on [Wikipedia](#)

**Using Euler's Formulas to derive Fourier Series approximations of a square wave 'by hand'**

or, it's not that scary! Start with  $P = 2\pi$

$$x(t) \approx a_0 + \sum_{n=1}^N \left( a_n \cos \left( \frac{2\pi nt}{P} \right) + b_n \sin \left( \frac{2\pi nt}{P} \right) \right)$$

$$a_0 = \frac{1}{P} \int_{-P/2}^{+P/2} x(t) dt$$

$$a_n = \frac{2}{P} \int_{-P/2}^{+P/2} x(t) \cos \left( \frac{2\pi nt}{P} \right) dt \quad n > 0$$

$$b_n = \frac{2}{P} \int_{-P/2}^{+P/2} x(t) \sin \left( \frac{2\pi nt}{P} \right) dt \quad n > 0$$

$$x(t) = \begin{cases} +1 & 0 < t < \pi \\ -1 & -\pi < t < 0 \end{cases}$$

$$x(t + 2\pi \times n) = x(t) \quad \forall n \in \mathbb{N}$$

### Computing $a_0$

$$a_0 = \frac{1}{P} \int_{-P/2}^{+P/2} x(t) dt \quad \text{where } P = 2\pi$$

Breaking up the integral into two parts, we get

$$\begin{aligned} &= \frac{1}{2\pi} \left( \int_{-\pi}^0 x(t) dt + \int_0^{+\pi} x(t) dt \right) \\ &= \frac{1}{2\pi} \left( \int_{-\pi}^0 (-k) dt + \int_0^{+\pi} (+k) dt \right) \\ &= \frac{1}{2\pi} \left( [-kt]_{-\pi}^0 + [kt]_0^{+\pi} \right) \\ &= \frac{1}{2\pi} \left( [0 - (-k \times (-\pi))] + (k\pi - 0) \right) \\ &= \frac{1}{2\pi} (-k\pi + k\pi) = 0 \end{aligned}$$

- We can also see that  $a_0$  must be zero because the area under the graph of  $x(t)$  is zero when computed from  $-\pi$  to  $+\pi$ .

### Computing $a_n$

$$\begin{aligned} a_n &= \frac{2}{P} \int_{-P/2}^{+P/2} x(t) \cos \left( \frac{2\pi nt}{P} \right) dt \quad \text{where } P = 2\pi \\ &= \frac{2}{2\pi} \int_{-\pi}^{+\pi} x(t) \cos \left( \frac{2\pi nt}{2\pi} \right) dt = \frac{1}{\pi} \int_{-\pi}^{+\pi} x(t) \cos(nt) dt \end{aligned}$$

Breaking up the integral into two parts and dropping the  $(t)$  notation

$$\begin{aligned}
&= \frac{1}{\pi} \left( \int_{-\pi}^0 x \cos ntdt + \int_0^{+\pi} x \cos ntdt \right) \\
&= \frac{1}{\pi} \left( \int_{-\pi}^0 (-1) \cos ntdt + \int_0^{+\pi} (+1) \cos ntdt \right) \\
&= \frac{1}{\pi} \left( \left[ -\frac{\sin nt}{n} \right]_{-\pi}^0 + \left[ \frac{\sin nt}{n} \right]_0^{+\pi} \right) \\
&= \frac{1}{n\pi} ([-\sin 0 - (-\sin(-n\pi))] + [\sin n\pi - \sin 0])
\end{aligned}$$

- Note that  $\sin x = 0$  at  $x = 0$ ,  $x = \pi$ , and indeed at  $x = n\pi$  for all integer  $n$ . So we learn that  $a_n = 0$  for all  $n$ .

### Computing $b_n$

$$\begin{aligned}
b_n &= \frac{2}{P} \int_{-P/2}^{+P/2} x(t) \sin\left(\frac{2\pi nt}{P}\right) dt \quad \text{where } P = 2\pi \\
&= \frac{2}{2\pi} \int_{-\pi}^{+\pi} x(t) \sin\left(\frac{2\pi nt}{2\pi}\right) dt = \frac{1}{\pi} \int_{-\pi}^{+\pi} x(t) \sin(nt) dt
\end{aligned}$$

Breaking up the integral into two parts and dropping the  $(t)$  notation

$$\begin{aligned}
&= \frac{1}{\pi} \left( \int_{-\pi}^0 x \sin ntdt + \int_0^{+\pi} x \sin ntdt \right) \\
&= \frac{1}{\pi} \left( \int_{-\pi}^0 (-1) \sin ntdt + \int_0^{+\pi} (+1) \sin ntdt \right) \\
&= \frac{1}{\pi} \left( \left[ \frac{1}{n} \cos nt \right]_{-\pi}^0 + \left[ -\frac{1}{n} \cos nt \right]_0^{+\pi} \right) \\
&= \frac{1}{\pi} \left( \left[ \frac{\cos nt}{n} \right]_{-\pi}^0 + \left[ -\frac{\cos nt}{n} \right]_0^{+\pi} \right) \\
&= \frac{1}{n\pi} (\cos 0 - \cos(-n\pi) + (-\cos n\pi - (-\cos 0))) \\
&= \frac{1}{n\pi} (\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0)
\end{aligned}$$

- Using the fact that  $\cos 0 = 1$  and  $\cos(-x) = \cos x$ ,

$$\implies b_n = \frac{2}{n\pi} (1 - \cos n\pi)$$

- Recall that  $\cos n\pi = \begin{cases} -1 & n \text{ odd} \\ +1 & n \text{ even} \end{cases}$

$$\Rightarrow b_n = \frac{2}{n\pi} (1 - \cos n\pi) = \begin{cases} \frac{2}{n\pi} (1 - (-1)) & n \text{ odd} \\ \frac{2}{n\pi} (1 - (+1)) & n \text{ even} \end{cases}$$

$$\Rightarrow b_n = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

### The full Fourier Series for a square wave

The Fourier Series representation of the square wave with amplitude 1 and period  $2\pi$ , i.e., of the function

$$x(t) = \begin{cases} +1 & 0 < t < \pi \\ -1 & -\pi < t < 0 \end{cases}$$

$$x(t + 2\pi \times n) = x(t) \quad \forall n \in \mathbb{N}$$

is therefore

$$x(t) = \sum_{n=1}^{\infty} (b_n \sin nt), \quad b_n = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \quad (1)$$

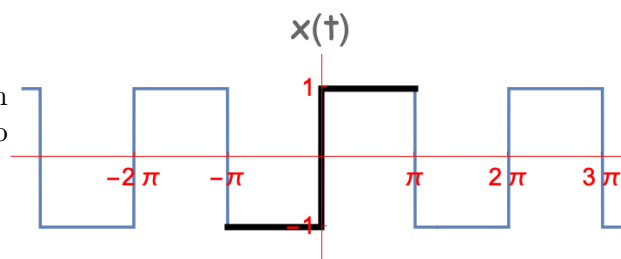
i.e.,  $x$  equals the infinite sum

$$x(t) = \frac{4}{\pi} \left( \frac{1}{1} \sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \dots \right)$$

where we use the term **partial sum up to  $N$**  to mean Equation 1 carried out up to  $n = N$ .

### Even and Odd Functions

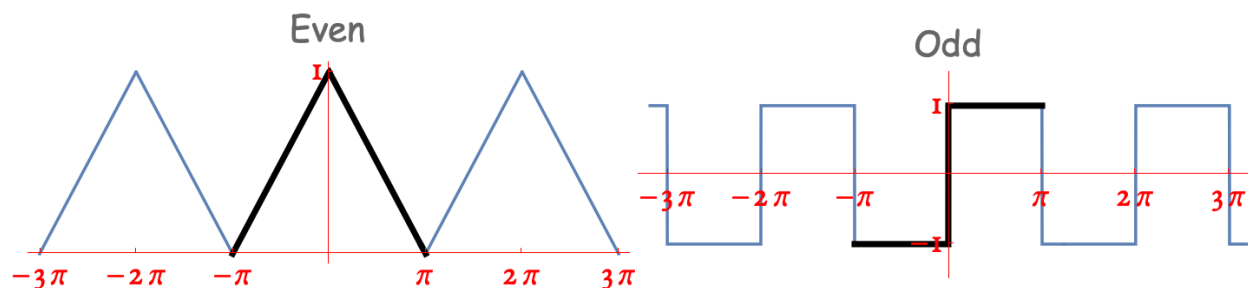
We notice that the Fourier Series for the function  $x(t)$  shown here contains only sin terms and no cos terms.



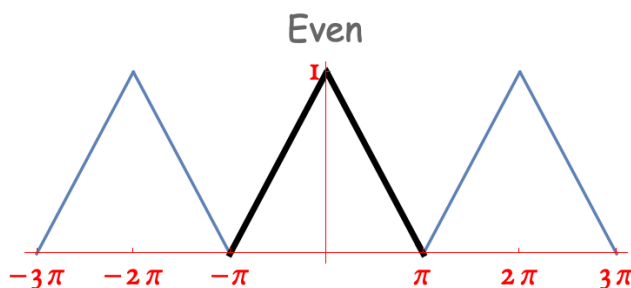
- This is because  $x(t)$  is an **odd function**.
- An **odd function**  $f(x)$  is a function for which  $f(x) = -f(-x)$  for all  $x$ .
- Conversely, an **even function**  $f(x)$  is a function for which  $f(x) = f(-x)$  for all  $x$ .

- Odd functions have Fourier Series that contain only sin terms
- Even functions have Fourier Series that contain only cos terms
- If a function is neither even nor odd, its Fourier Series contains both sin and cos terms.

### Visualizing Even vs odd functions



### Calculating coefficients for another Fourier Series



Show that a mathematical description of the given triangle wave with fundamental period  $2\pi$  is

$$x(t) = \begin{cases} \frac{t-\pi}{\pi} & -\pi < t < 0 \\ \frac{\pi-t}{\pi} & 0 < t < +\pi \end{cases}$$

Then, show by using piecewise integration of the Euler Formulas that

- $a_n = \frac{4}{n^2\pi^2}$  for odd values of  $n > 0$ ,
- $a_n = 0$  for even values of  $n$
- $a_0 = 1/2$ .

The infinite Fourier series for this function is

$$x(t) = \frac{4}{\pi^2} \left( \cos t + \frac{\cos 3t}{9} + \frac{\cos 5t}{25} + \dots \right)$$

## In-class notes

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A periodic function  $x(t)$  with period  $P$  is equal to the convergent infinite series

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{2\pi n}{P}t\right) + b_n \sin\left(\frac{2\pi n}{P}t\right) \right)$$

where

$$a_0 = \frac{1}{P} \int_{-P/2}^{P/2} x(t) dt$$

$$a_n = \frac{2}{P} \int_{-P/2}^{P/2} x(t) \cos\left(\frac{2\pi n}{P}t\right) dt$$

$$b_n = \frac{2}{P} \int_{-P/2}^{P/2} x(t) \sin\left(\frac{2\pi n}{P}t\right) dt$$

$x(t)$  is

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## In-class notes

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$$x(t) = \begin{cases} -k & \text{if } -\frac{p}{2} < x < 0 \\ k & \text{if } 0 < x < \frac{p}{2} \end{cases}$$
$$a_0 = \frac{1}{p} \int_{-\frac{p}{2}}^{\frac{p}{2}} x(t) dt$$
$$a_0 = \frac{1}{p} \int_{-\frac{p}{2}}^0 (-k) dt + \frac{1}{p} \int_0^{\frac{p}{2}} k dt$$
$$a_0 = \frac{-k}{p} t \Big|_{-\frac{p}{2}}^0 + \frac{k}{p} t \Big|_0^{\frac{p}{2}}$$
$$a_0 = 0 - \frac{k}{p} \left(\frac{p}{2}\right) + \frac{k}{p} \left(\frac{p}{2}\right) - 0$$
$$a_0 = -\frac{k}{2} + \frac{k}{2} \quad \boxed{a_0 = 0}$$

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$$a_n = \frac{2}{P} \int_{-P/2}^{P/2} x(t) \cos\left(\frac{2\pi n}{P} t\right) dt$$

$$a_n = \frac{2}{P} \int_{-P/2}^0 (-k) \cos\left(\frac{2\pi n}{P} t\right) dt + \frac{2}{P} \int_0^{P/2} k \cos\left(\frac{2\pi n}{P} t\right) dt$$

$$a_n = \frac{2}{P} (-k) \left(\frac{P}{2\pi n}\right) \sin\left(\frac{2\pi n}{P} t\right) \Big|_{-P/2}^0 + \dots$$

$$\frac{2}{P} k \frac{P}{2\pi n} \sin\left(\frac{2\pi n}{P} t\right) \Big|_0^{P/2}$$

$$a_n = \frac{-k}{\pi n} \sin\left(\frac{2\pi n}{P} t\right) \Big|_{-P/2}^0 + \frac{k}{\pi n} \sin\left(\frac{2\pi n}{P} t\right) \Big|_0^{P/2}$$

$$a_n = 0 + \frac{k}{\pi n} \sin\left(\frac{2\pi n}{P} \cdot \left(-\frac{P}{2}\right)\right) + \frac{k}{\pi n} \sin\left(\frac{2\pi n}{P} \cdot \frac{P}{2}\right) - 0$$

$$a_n = \frac{k}{\pi n} \sin(-\pi n) + \frac{k}{\pi n} \sin(\pi n)$$

$\sin(-x) = -\sin(x)$

$$a_n = \frac{-k}{\pi n} \sin(\pi n) + \frac{k}{\pi n} \sin(\pi n)$$

$a_n = 0$

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$$b_n = \frac{2}{P} \int_{-P/2}^{P/2} x(t) \sin\left(\frac{2\pi n}{P} t\right) dt$$

$$b_n = \frac{2}{P} \int_{-P/2}^0 (-K) \sin\left(\frac{2\pi n}{P} t\right) dt + \frac{2}{P} \int_0^{P/2} K \sin\left(\frac{2\pi n}{P} t\right) dt$$

$$b_n = \frac{2}{P} (-K) \left(\frac{P}{2\pi n}\right) \left(-\cos\left(\frac{2\pi n}{P} t\right)\right) \Big|_{-P/2}^0 + \dots$$

$$\frac{2}{P} K \left(\frac{P}{2\pi n}\right) \left(-\cos\left(\frac{2\pi n}{P} t\right)\right) \Big|_0^{P/2}$$

$$b_n = \frac{K}{\pi n} \cos\left(\frac{2\pi n}{P} t\right) \Big|_{-P/2}^0 + \frac{-K}{\pi n} \cos\left(\frac{2\pi n}{P} t\right) \Big|_0^{P/2}$$

$$b_n = \frac{K}{\pi n} - \frac{K}{\pi n} \cos\left(\frac{2\pi n}{P} \cdot \left(-\frac{P}{2}\right)\right) + \frac{-K}{\pi n} \cos\left(\frac{2\pi n}{P} \cdot \frac{P}{2}\right) + \frac{K}{\pi n}$$

$$b_n = \frac{K}{\pi n} - \frac{K}{\pi n} \cos(\pi n) + \frac{-K}{\pi n} \cos(\pi n) + \frac{K}{\pi n}$$

$\cos(-x) = \cos x$

$$b_n = \frac{K}{\pi n} (2 - 2\cos(\pi n))$$

$$b_n = \frac{2K}{\pi n} (1 - \cos(\pi n))$$

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$$b_n = \frac{2k}{\pi n} (1 - \cos(\pi n))$$

$n$  is even ...

$$b_n = \frac{2k}{\pi n} (1 - 1) = 0$$

$n$  is odd ...

$$b_n = \frac{2k}{\pi n} (1 - (-1)) = \frac{4k}{\pi n}$$

$$x(t) = \cancel{a_0} + \sum_{n=1}^{\infty} \left( \cancel{a_n \cos\left(\frac{2\pi n}{p} t\right)} + b_n \sin\left(\frac{2\pi n}{p} t\right) \right)$$

$$x(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{p} t\right)$$

where  $b_n = \begin{cases} 0, & n \text{ is even} \\ \frac{4k}{\pi n}, & n \text{ is odd} \end{cases}$

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$$x(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{P} t\right)$$

where  $b_n = \begin{cases} 0, & n \text{ is even} \\ \frac{4k}{\pi n}, & n \text{ is odd} \end{cases}$

For  $k=1$ ,  $P=2\pi$ ,  $N=5$

$$x(t) \approx \frac{4}{\pi} \left( \sin(t) + \frac{1}{3} \sin(3t) + \frac{1}{5} \sin(5t) \right)$$

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$x(t) = \sin(3t)$

What is the value  $T$  such  
that  $g(t+nT) = g(t)$   
for all  $n \in \text{integers}$ ?

$y(t) = \sin(\omega t + \phi)$

angular frequency  $\omega$  phase offset  $\phi$

$\omega = 2\pi f = \frac{2\pi}{T} = 3$

$T = \frac{2\pi}{3} \approx 2.1$

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