Appendix for Final Exam

ENGR 12, Spring 2026.

Appendix

Units

In the SI system,

- The units of resistance are Ohms,
- The units of capacitance are Farads, and
- The units of inductance are Henrys.

Electrical quantities in the base units of kilograms, seconds, meters, and Amperes are not always very useful. However, it is useful to note that:

- Multiplying Ohms by Farads gives seconds, and
- Multiplying Henrys by Farads gives seconds squared.

First-order systems

A first-order system with transfer function $\frac{1}{\tau s+1}$ has time constant τ. In connection with time constants, the following table may be useful.

The free response of a first-order system $\tau\dot{x}+x=f(t)=0$ initialized at $x=x_0$ at integer multiples of τ is given by the following table.

Time	Value	Decimal	Percent
$t=0\tau$	$x=x_0e^{-0}$	$x=1.0x_0$	100%
$t=1\tau$	$x=x_0e^{-1}$	$x=0.3678x_0$	36.7%
$t=2\tau$	$x=x_0e^{-2}$	$x=0.1353x_0$	13.5%
$t=3\tau$	$x=x_0e^{-3}$	$x=0.0497x_0$	4.9%
$t=4\tau$	$x=x_0e^{-4}$	$x=0.0183x_0$	1.8%
$t=5\tau$	$x=x_0e^{-4}$	$x=0.0067x_0$	0.6%

Second-order systems

The second-order differential equation $m\ddot{x} + b\dot{x} + kx = f(t)$ can be represented using the transfer function $\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ if the input is $f(t)$ and the output kx, where the undamped natural frequency is

$$\omega_n = \sqrt{k/m}$$

and the damping ratio is

$$\zeta = \frac{b}{2\sqrt{mk}}.$$

The solution to the differential equation

$$m\ddot{x} + b\dot{x} + kx = 0$$

has three possible forms depending on whether the roots of the characteristic polynomial are:

1. Real and distinct: $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$, where λ_1 and λ_2 are the two roots of the characteristic polynomial
2. Complex: $x(t) = e^{rt}\left(c_1 \cos\omega_d t + c_2 \sin\omega_d t\right)$ where r and ω_d are the real and imaginary parts of the roots of the characteristic polynomial
3. Real and repeated: $x(t) = \left(c_1 + c_2 t\right) e^{\lambda t}$ where λ is the repeated root.

For underdamped systems,

- The damped natural frequency is $\omega_d = \omega_n\sqrt{1-\zeta^2}$
- The resonant frequency is $\omega_d = \omega_n\sqrt{1-2\zeta^2}$
- The time constant is
$$\frac{1}{\zeta\omega_n}$$

Fourier Series

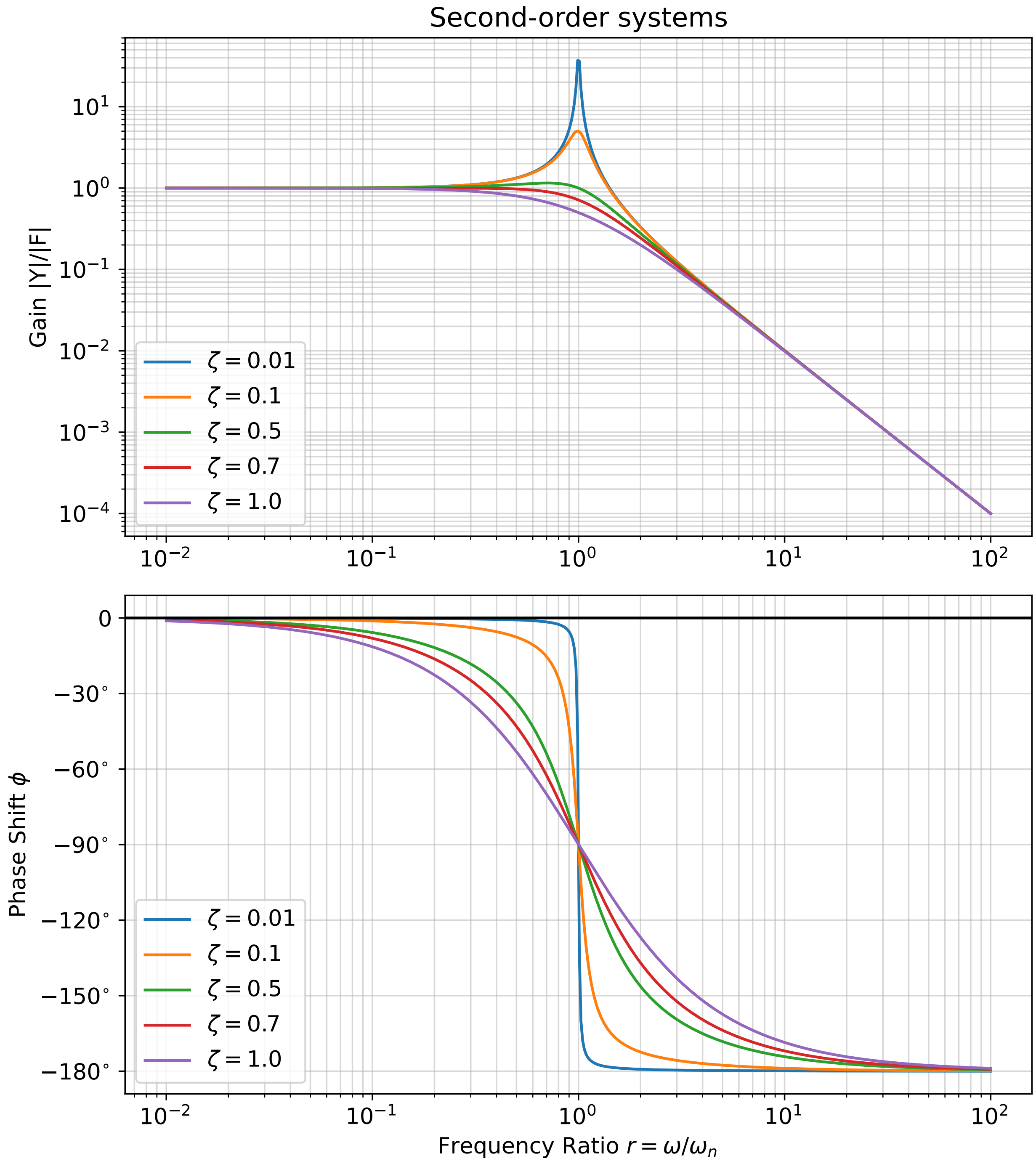
A periodic function $x(t)$ with period P is equal to the convergent infinite series

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2\pi n t}{P}\right) + b_n \sin\left(\frac{2\pi n t}{P}\right)\right)$$
$$x(t) \approx a_0 + \sum_{n=1}^{N}\left(a_n \cos\left(\frac{2\pi n t}{P}\right) + b_n \sin\left(\frac{2\pi n t}{P}\right)\right)$$

for a_n, b_n given by the **Euler Formulas**:

$$a_0 = \frac{1}{P}\int_{-P/2}^{+P/2} x(t)dt$$
$$a_n = \frac{2}{P}\int_{-P/2}^{+P/2} x(t)\cos\left(\frac{2\pi n t}{P}\right)dt \quad n > 0$$
$$b_n = \frac{2}{P}\int_{-P/2}^{+P/2} x(t)\sin\left(\frac{2\pi n t}{P}\right)dt \quad n > 0$$

Bode Plots of Second-order systems



Partial Fraction Expansion

Consider the rational function

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + ... + b_0 s^0}{s^n + a_{n-1} s^{n-1} + ... + a_0 s^0}$$

Distinct Poles

If $D(s)$ has n distinct roots, then we can use partial fractions to expand

$$F(s) = \frac{N(s)}{D(s)} = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + ... + \frac{A_n}{s - s_n}$$

where

$$A_i = \lim_{s \to s_i} \left[(s - s_i) F(s)\right]$$

Repeated Poles

Let $F(s) = \frac{N(s)}{D(s)}$ where $D(s)$ has 2 repeated poles and $n - 2$ distinct poles. Then we can use partial fractions to expand

$$F(s) = \frac{N(s)}{D(s)} = \frac{A_{11}}{(s - s_1)^2} + \frac{A_{12}}{s - s_1} + \frac{A_3}{s - s_3} + ... + \frac{A_n}{s - s_n}$$

where the coefficients $A_3, ...A_n$ can be found using a similar process as before:

$$A_i = \lim_{s \to s_i} \left[(s - s_i) F(s)\right]$$

and we have formulas for the numerators of the repeated poles:

$$A_{11} = \lim_{s \to s_1} \left[(s - s_1)^2 F(s)\right]$$

$$A_{12} = \lim_{s \to s_1} \left(\frac{d}{ds} \left[(s - s_1)^2 F(s)\right]\right)$$

Complex Poles

If $D(s)$ has two complex roots, then $F(s)$ can be expanded as

$$F(s) = \frac{K_1}{s - (r + i\omega)} + \frac{K_2}{s - (r - i\omega)}$$

where ω is the imaginary part of the complex roots and r the real part.

$$K_1 = \lim_{s \to r + i\omega} \left[(s - (r + i\omega)) F(s)\right]$$

$$K_2 = \lim_{s \to r - i\omega} \left[(s - (r - i\omega)) F(s)\right]$$

Laplace Transfroms

The Laplace Transform of a function $x(t)$ is **defined** as the following function of s:

$$\lim_{T\to\infty}\int_0^T x(t)e^{-st}dt \tag{1}$$

This is a function of s, not a function of t. We give the expression in Equation 1 the name $X(s)$.

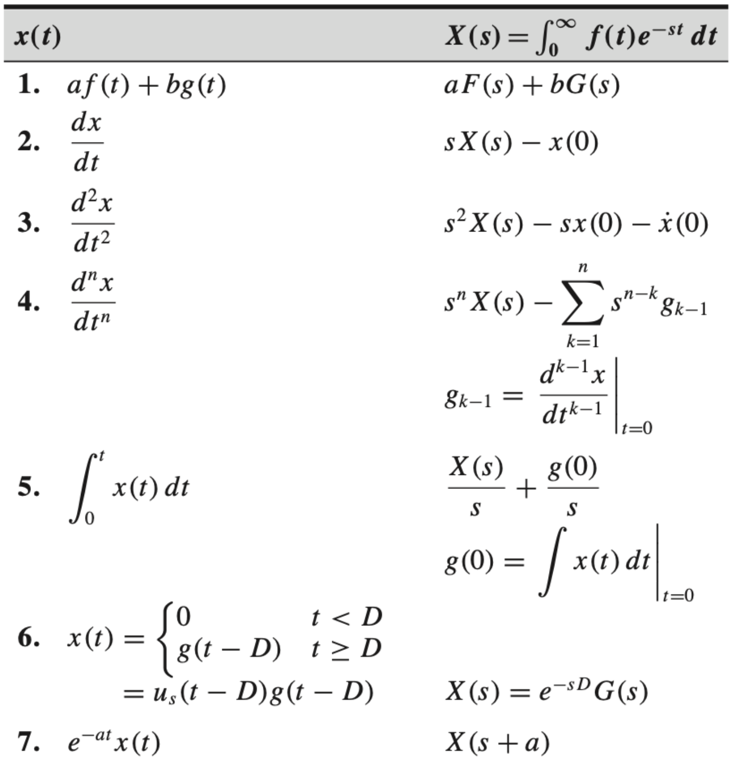
$$X(s) = \boxed{\lim_{T\to\infty}\int_0^T x(t)e^{-st}dt} = \int_0^\infty x(t)e^{-st}dt$$

$$x(t) \to \text{Laplace Transform} \to X(s)$$

$$x(t) \to \mathcal{L}[\cdot] \to X(s)$$

$$\boxed{\mathcal{L}[x(t)] = X(s)}$$

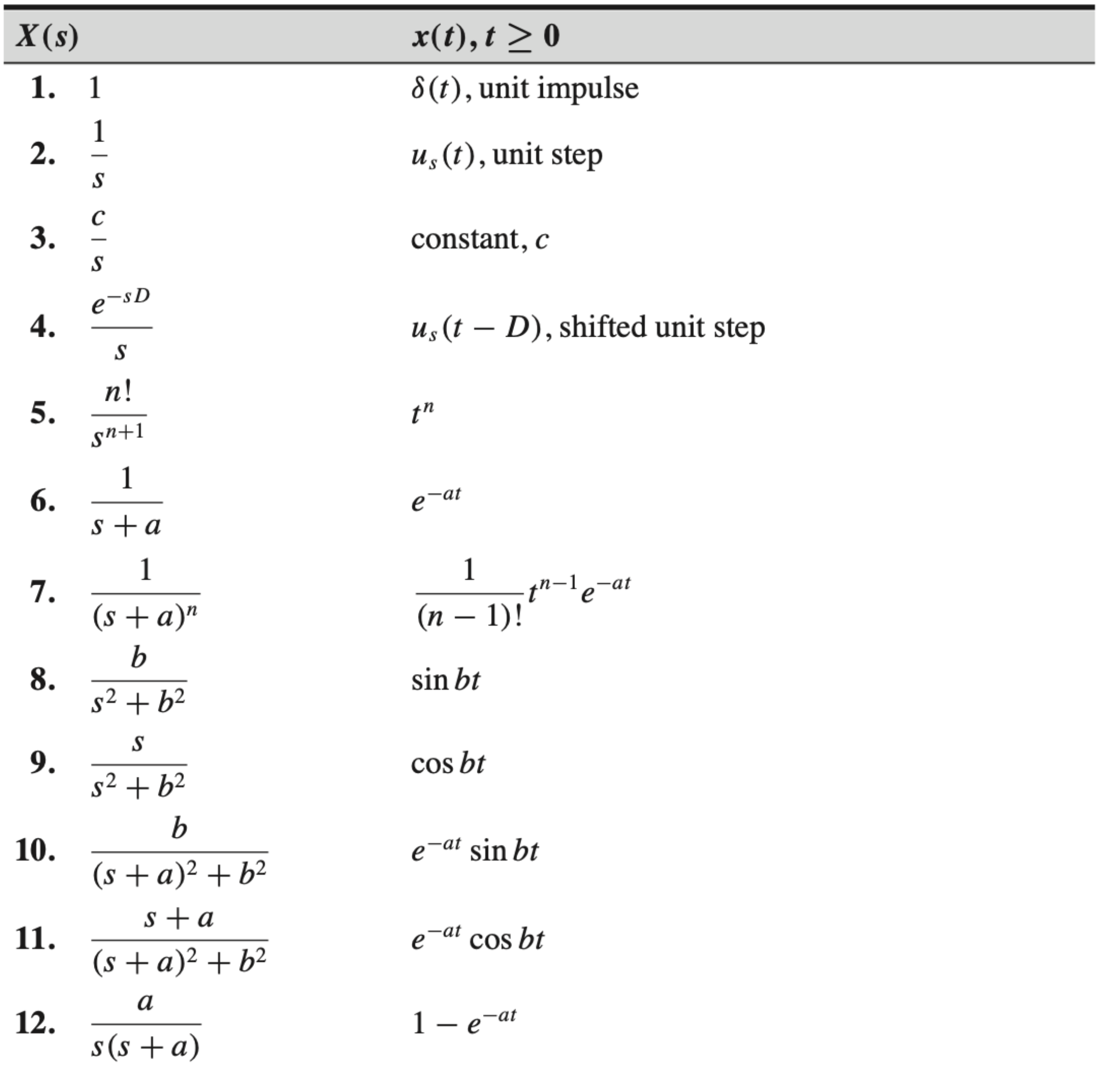
Table 3.3.2 Properties of the Laplace transform.



$x(t)$	$X(s) = \int_0^\infty f(t)e^{-st}\,dt$
1. $af(t) + bg(t)$	$aF(s) + bG(s)$
2. $\dfrac{dx}{dt}$	$sX(s) - x(0)$
3. $\dfrac{d^2x}{dt^2}$	$s^2X(s) - sx(0) - \dot{x}(0)$
4. $\dfrac{d^nx}{dt^n}$	$s^nX(s) - \sum_{k=1}^{n} s^{n-k}g_{k-1}$ $g_{k-1} = \left.\dfrac{d^{k-1}x}{dt^{k-1}}\right\rvert_{t=0}$
5. $\int_0^t x(t)\,dt$	$\dfrac{X(s)}{s} + \dfrac{g(0)}{s}$ $g(0) = \left.\int x(t)\,dt\right\rvert_{t=0}$
6. $x(t) = \begin{cases} 0 & t < D \\ g(t-D) & t \geq D \end{cases}$ $= u_s(t-D)g(t-D)$	$X(s) = e^{-sD}G(s)$
7. $e^{-at}x(t)$	$X(s+a)$

Laplace Tables

Table 3.3.1 Table of Laplace transform pairs.



$X(s)$	$x(t), t \geq 0$
1. 1	$\delta(t)$, unit impulse
2. $\frac{1}{s}$	$u_s(t)$, unit step
3. $\frac{c}{s}$	constant, c
4. $\frac{e^{-sD}}{s}$	$u_s(t-D)$, shifted unit step
5. $\frac{n!}{s^{n+1}}$	t^n
6. $\frac{1}{s+a}$	e^{-at}
7. $\frac{1}{(s+a)^n}$	$\frac{1}{(n-1)!}t^{n-1}e^{-at}$
8. $\frac{b}{s^2+b^2}$	$\sin bt$
9. $\frac{s}{s^2+b^2}$	$\cos bt$
10. $\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
11. $\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
12. $\frac{a}{s(s+a)}$	$1-e^{-at}$