ENGR 21:
Computer Engineering Fundamentals

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Lecture 13
Tuesday, October 21, 2025

Solving Linear Systems on a computer

(plus a tiny bit of linear algebra)

Linear Systems of the form $\mathbf{Ax} = \mathbf{b}$ arise in many engineering systems

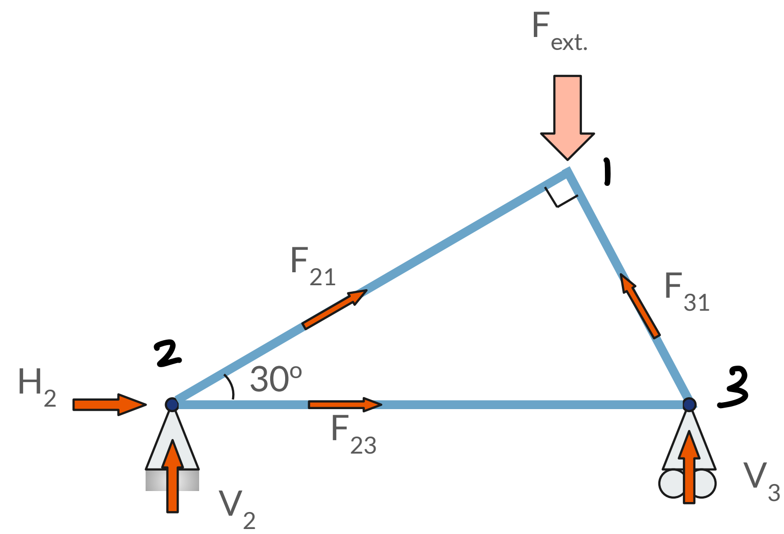
$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ A_{41} & A_{42} & A_{43} \\ A_{51} & A_{52} & A_{53} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

$$\boxed{\begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 &= b_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 &= b_2 \\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3 &= b_3 \\ A_{41}x_1 + A_{42}x_2 + A_{43}x_3 &= b_4 \\ A_{51}x_1 + A_{52}x_2 + A_{53}x_3 &= b_5 \end{aligned}}$$

A_{ij} – i^{th} row, j^{th} column

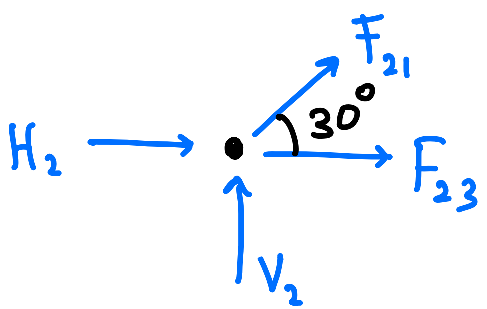
Motivating Example (1 of 2) from ENGR 6

Find the reaction forces H_2, V_2, and V_3 and the internal forces



$\sum F_x = 0$ at 1, 2, 3

$\sum F_y = 0$ at 1, 2, 3



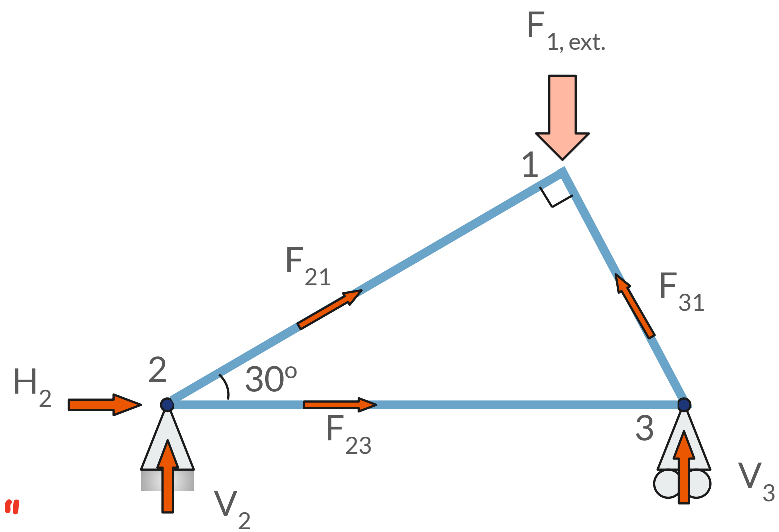
Horizontal force balance for joint 2:

$1\,H_2 + 0\,V_2 + 0\,V_3 + (\cos 30^\circ)\,F_{21} + 1\,F_{23} + 0\,F_{31} = \{$ external horiz. force at 2 $\}$

$\square H_2 + \square V_2 + \square V_3 + \square F_{21} + \square F_{23} + \square F_{31} = \ldots$

Motivating Example (1 of 2) from ENGR 6

Find the reaction forces H_2, V_2, and V_3 and the internal forces



unknown.

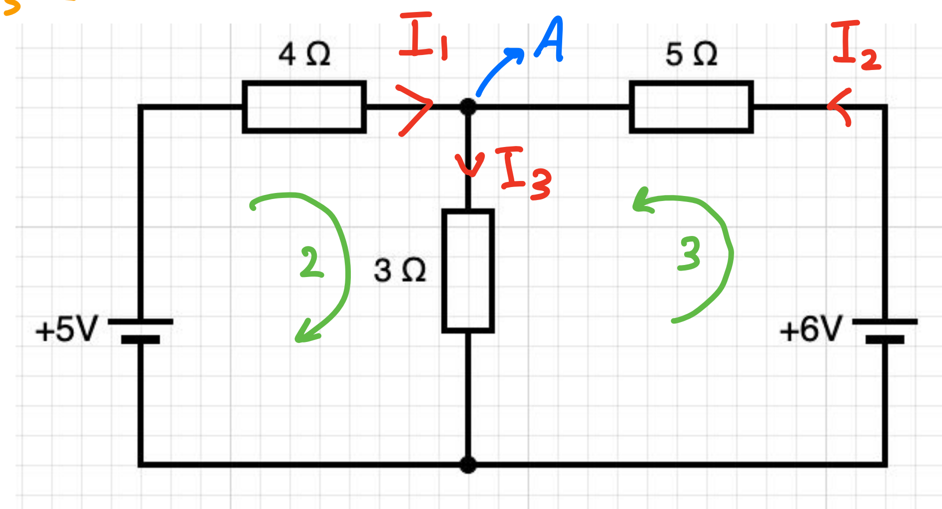
$$\begin{array}{l} \text{horz. 2} \\ \\ \text{vert. 1} \\ \\ \end{array} \overbrace{\begin{bmatrix} 1 & 0 & 0 & \cos 30^\circ & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \sin 30^\circ & 0 & \sin 60^\circ \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}}^{\text{"A"}} \overbrace{\begin{bmatrix} H_2 \\ V_2 \\ V_3 \\ F_{21} \\ F_{23} \\ F_{31} \end{bmatrix}}^{x} = \overbrace{\begin{bmatrix} 0 \\ \vdots \\ F_{1,\text{ext}} \\ \vdots \end{bmatrix}}^{\text{"b"}}$$

2 out of 6 equations

Motivating Example (2 of 2) from ENGR 11

$I_1 + I_2 - I_3 = 0$

Find the currents in this circuit



1) Sum of currents at node A

$$0 + 1I_1 + 1I_2 - 1I_3 = 0$$

2) Loop 2 :

$$+5 - 4I_1 + 0I_2 - 3I_3 = 0$$

3) Loop 3 :

$$+6 + 0I_1 - 5I_2 - 3I_3 = 0$$

$$\overbrace{\begin{bmatrix} -4 & 0 & -3 \\ 0 & -5 & -3 \\ 1 & 1 & -1 \end{bmatrix}}^{\text{known} \rightarrow A} \overbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}}^{\text{unknown } x} = \overbrace{\begin{bmatrix} -5 \\ -6 \\ 0 \end{bmatrix}}^{\text{known } b}$$

Solving systems of the kind Ax = b

$A x_1 = b_1 \quad : \quad x_1 = A^{-1} b_1 \ \ldots$

$A x_2 = b_2 \quad \quad x_2 = A^{-1} b_2 \ \ldots$

$$\boxed{Ax = b \Rightarrow x = A^{-1} b}$$

- Equivalent to "inverting the matrix"
- A: the "system matrix"
- Once you have found A^{-1}, you can apply it to many different right-hand-side vectors b.

A is often too large to invert. e.g. N ≈ 10^6

$$A^{-1} \cdot A = \text{identity matrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left(2^{-1} \cdot 2 = 1\right)$$

What if the number of equations is not equal to the number of unknowns?

- # Unknowns > # Equations — : need additional constraints
- # Equations > # Unknowns — : problem is 'overdetermined'
- If # Equations = # Unknowns:
  - . if one equation depends on another, or reflects the same information as another equation, you can't double count.
  - . need equations to be 'linearly independent'

Gaussian Elimination followed by substitution

- Systematic way of solving simultaneous equations

$$\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$2x_1 + 3x_2 = 4 \quad (1)$$

$$-x_1 + 2x_2 = 5 \quad (2)$$

Replace (2) with an equation that only has x_2 as unknown.

eq.(1) + 2 × eq.(2) ⟶ new eq.(2)

$$\begin{array}{rrrl} & 2x_1 & +3x_2 & = 4 \\ + & -2x_1 & +4x_2 & = 10 \\ \hline & 0x_1 & +7x_2 & = 14 \quad (2^*) \end{array}$$

if eqns were

$$2x_1 + 3x_2 = 4$$

$$4x_1 + 6x_2 = 8$$

linearly dependent! (Bad)

⟶

- Solve eq.(2) for x_2
- substitute x_2 into (1)
- Solve eq.(1) for x_1

Teaching a computer to perform elimination & substitution

1. Start with Ax = b

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

2. Build an 'augmented matrix' using A and b

$$\left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & b_1 \\ A_{21} & A_{22} & A_{23} & b_2 \\ A_{31} & A_{32} & A_{33} & b_3 \end{array}\right]$$

Augmented matrix

3. Perform 'row operations' until you are left with an augmented matrix of the form shown here, called 'row echelon form'.

$$\left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & b_1 \\ 0 & A_{22} & A_{23} & b_2 \\ 0 & 0 & A_{33} & b_3 \end{array}\right]$$

'row echelon form'

4. Solve equations starting from the bottom, working your way up

What are 'row operations' ?

There are three types of possible row operations. Typically, we do these operations on the augmented matrix.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \longrightarrow \begin{bmatrix} j & k & l \\ 0 & m & n \\ 0 & 0 & p \end{bmatrix}$$

1. Row switching.
2. Multiply a row by a non-zero constant
3. Add one row to another

The Gaussian (forward) Elimination Algorithm

$$\left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & b_1 \\ A_{21} & A_{22} & A_{23} & b_2 \\ A_{31} & A_{32} & A_{33} & b_3 \end{array}\right]$$

1) Don't change row 1.

2) Let updated row 2 = row 2 − $\overset{\text{multiplier}}{\boxed{\dfrac{A_{21}}{A_{11}}}}$ × row 1

$$\left[\begin{array}{ccc|c} A_{21} & A_{22} & A_{23} & b_2 \end{array}\right] - \frac{A_{21}}{A_{11}} \times \left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & b_1 \end{array}\right]$$: new row 2.

$$\left[\begin{array}{ccc|c} \cancelto{0}{A_{21} - \boxed{\frac{A_{21}}{A_{11}}} \times A_{11}}, & A_{22} - \boxed{\frac{A_{21}}{A_{11}}} \times A_{12}, & A_{23} - \boxed{\frac{A_{21}}{A_{11}}} \times A_{13}, & b_2 - \boxed{\frac{A_{21}}{A_{11}}} \times b_1 \end{array}\right]$$

3) Let updated row 3 = row 3 − $\underset{\text{multiplier}}{\boxed{\dfrac{A_{31}}{A_{11}}}}$ × row 1

$$\left[\begin{array}{cccc} A_{11} & A_{12} & A_{13} & b_1 \\ 0 & A_{22} & A_{23} & b_2 \\ A_{31} & A_{32} & A_{33} & b_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} \cancelto{0}{A_{31} - \boxed{\frac{A_{31}}{A_{11}}}\, A_{11}}, & A_{32} - \boxed{\frac{A_{31}}{A_{11}}} \cdot A_{12}, & A_{33} - \boxed{\frac{A_{31}}{A_{11}}} \times A_{13}, & b_3 - \boxed{\frac{A_{31}}{A_{11}}} \times b_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & b_1 \\ 0 & A_{22} & A_{23} & b_2 \\ 0 & A_{32} & A_{33} & b_3 \end{array}\right]$$

The Gaussian (forward) Elimination Algorithm

4) Let updated row 3 = row 3 $- \boxed{\frac{A_{32}}{A_{22}}} \times$ row 2

$$\Big[\underbrace{A_{31} - \boxed{\frac{A_{32}}{A_{22}}} \times A_{21}}_{\text{already zero.}},\ \cancel{A_{32} - \boxed{\frac{A_{32}}{A_{22}}} \cdot A_{22}} \rightarrow 0,\ A_{33} - \boxed{\frac{A_{32}}{A_{22}}} \cdot A_{23},\ \Big|\ b_3 - \boxed{\frac{A_{32}}{A_{22}}} \cdot b_2 \Big]$$

$$\left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & b_1 \\ 0 & A_{22} & A_{23} & b_2 \\ 0 & 0 & A_{33} & b_3 \end{array}\right] : \text{equivalent to} \left[\begin{array}{rcrcrcl} A_{11}x_1 & + & A_{12}x_2 & + & A_{13}x_3 & = & b_1 \\ & & A_{22}x_2 & + & A_{23}x_3 & = & b_2 \\ & & & & A_{33}x_3 & = & b_3 \end{array}\right]$$

The backward substitution algorithm

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

We start from the last row ...

$$x_3 = b_3 / A_{33}$$

$$x_2 = \frac{b_2 - A_{23} \cdot \overbrace{x_3}^{\text{known}}}{A_{22}}$$

$$x_1 = \frac{b_1 - A_{12} \overbrace{x_2}^{\text{known}} - A_{13} \overbrace{x_3}^{\text{known}}}{A_{11}}$$

$$x_k = \frac{1}{A_{kk}} \left(b_k - \sum_{j=k+1}^{n} A_{kj} x_j \right)$$

Practicing Gaussian Elimination + Backward Substitution by hand

- Which elements are eliminated first? What's the order?

$A_{21} \rightarrow A_{31} \rightarrow A_{41}$

$\rightarrow A_{32} \rightarrow A_{42}$

$\rightarrow A_{43}$

$$
\underbrace{\begin{bmatrix} 2x_1 + 1x_2 - 1x_3 + 2x_4 \\ 4x_1 + 5x_2 - 3x_3 + 6x_4 \\ -2x_1 + 5x_2 - 2x_3 + 6x_4 \\ 4x_1 + 11x_2 - 4x_3 + 8x_4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 5 \\ 9 \\ 4 \\ 2 \end{bmatrix}}_{b}
$$