ENGR 21:
Computer Engineering Fundamentals

Lecture 14
Thursday, October 23, 2025

Recall: Linear systems

- Goal: use a computer to numerically compute solutions of the equation $\boldsymbol{Ax} = \boldsymbol{b}$

n x n matrix

n x 1 vector

n x 1 vector

The inverse of a matrix and the solution to Ax=b

if there is $\boldsymbol{A}^{-1}$ such that $\boldsymbol{A}\boldsymbol{A}^{-1} = \mathbf{1}$ identity matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & & \ddots \\ 0 & & 1 & \\ \vdots & \ddots & & 1 \end{bmatrix}$$

then $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \implies \underbrace{\boldsymbol{A}^{-1}\boldsymbol{A}}\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$

$\implies \boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$

$A^{-1}A = \mathbf{1}$

$\boxed{\mathbf{1}x = x}$

if you know A^{-1} already, you don't need a numerical method to solve $Ax=b$

$x = A^{-1}b$

But is there a $\boldsymbol{A}^{-1}$?

Yes : solution exists

No : solution does not exist.

When systems of linear equations don't have a unique solution

- We usually expect n equations for n unknowns to have a solution. But consider:

$$\begin{aligned} 2x + y &= 3 \\ 4x + 2y &= 6 \end{aligned}$$

same information

effectively : 1 eqn. 2 unknowns

infinitely many solutions ✗

contradict each other

$$\begin{aligned} 2x + y &= 3 \\ 4x + 2y &= 0 \end{aligned}$$

inconsistent

No solution ✗

- The problem here is that the system matrix is singular.

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

The Determinant of a (square) matrix

`np.linalg.det`

- A scalar value that gives us information about a matrix. For 2x2 matrices:

$$\text{“}\det \mathbf{A}\text{”} = \text{“}|\mathbf{A}|\text{”} = A_{11} \times A_{22} - A_{12} \times A_{21} \xleftarrow{\det} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

- For 3x3 matrices:

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

- Let's calculate the determinant of the matrix from the previous problem:

$$\left| \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \right| = 2 \times 2 - 1 \times 4 = 0$$

The following statements are equivalent.

1) • There is a unique solution to the equation $\boldsymbol{Ax} = \boldsymbol{b}$
2) • The matrix A is nonsingular
3) • The determinant of A is nonzero
4) • A is invertible, i.e., there exists A^{-1}

if one is true,
they are all true
if one is false,
they are all false.

$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$

Why does all this matter?

- We are interested in developing and using numerical methods to solve systems of equations of the form $\boldsymbol{Ax} = \boldsymbol{b}$
- But before we do this, we must be sure that there **is** a solution for the computer to find.

$$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = ad - bc$$

Well-conditioned and ill-conditioned matrices

Sometimes, matrices are **almost singular**. What does this mean?

Calculate the determinant of the matrix representing the system of equations

$$\begin{bmatrix} 2x + y = 3 \\ 2x + y = 0 \end{bmatrix}$$

A det $A = 0$

B det $B = 0.001$

C det $C = 0.1$

D det $D = 10$

{ Small* determinant : almost singular } (A, B)

{ not-small determinant: not almost singular } (C, D)

$$2x + y = 3$$
$$2x + 1.001y = 0$$

determinant $= \boxed{0.002}$

*: small relative to the elements of the matrix.

$$\|\mathbf{A}\| = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}^2}$$

numpy.linalg.norm

$$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = ad - bc$$

Well-conditioned and ill-conditioned matrices

Sometimes, matrices are **almost singular**. What does this mean?

To quantify 'almost singular'-ness:

$\boxed{\text{norm of } \underset{\sim}{A}}$: "size of elements of A"

The det. of a matrix is small if $|\underset{\sim}{A}| \underbrace{\ll}_{\text{much less than}} \|\underset{\sim}{A}\|$

much less than, i.e. "order of magnitude less than"

condition number

$$\text{cond}(\underset{\sim}{A}) = \|\underset{\sim}{A}\| \times \|\underset{\sim}{A}^{-1}\|$$

if $\text{cond}(\underset{\sim}{A}) \approx 1$: $\underset{\sim}{A}$ is well-conditioned

if $\text{cond}(\underset{\sim}{A}) \gg 1$: $\underset{\sim}{A}$ is ill-conditioned

$\text{cond}(\underset{\sim}{A}) \longrightarrow \infty$ as $\underset{\sim}{A} \longrightarrow$ singular.

$$\|\mathbf{A}\| = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}^2}$$

$\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{b}$

Direct vs. Iterative methods of solving linear systems

Direct Methods

- e.g. Gaussian Elimination + Backward substitution
- Others:
  - LU Decomposition,
  - Matrix Inversion
- After a certain number of steps, you arrive at the correct solution.

Iterative Methods

- e.g. Gauss-Seidel method
- Others:
  - Method of steepest descent
- Each iteration improves your guess until your solution is good enough.

The iterative process of solving Ax=b (for any iterative method)

Start with a guess

Quit when good enough

$$\mathbf{x}^{(0)} \to \mathbf{x}^{(1)} \to \mathbf{x}^{(2)} \to \mathbf{x}^{(3)} \to \mathbf{x}^{(4)} \to \mathbf{x}^{(5)} \to \mathbf{x}^{(6)} \to \dots$$