ENGR 21:
Computer Engineering Fundamentals

Lecture 15
Tuesday, October 28, 2025

Announcing final project

Back to Linear Systems of equations

Direct vs. Iterative methods of solving linear systems

Direct Methods

- e.g. Gaussian Elimination + Backward substitution
- Others:
  - LU Decomposition,
  - Matrix Inversion
- After a certain number of steps, you arrive at the correct solution.

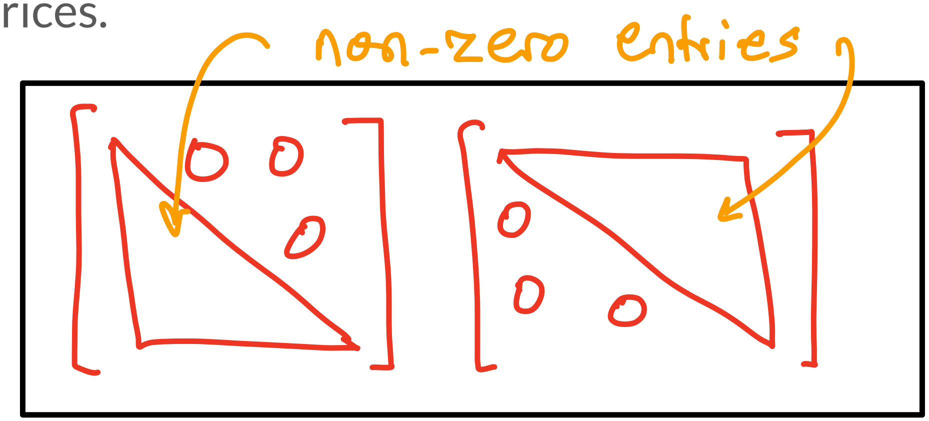
Iterative Methods

- e.g. Gauss-Seidel method
- Others:
  - Method of steepest descent
- Each iteration improves your guess until your solution is good enough.

Another Direct Method: for solving Ax=b: LU Decomposition

Suppose we know that A can be decomposed into the product of two matrices.

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1 & 2/3 \\ 1/3 & 2/3 & 1 \end{bmatrix} =$$



Lower Triangular Matrix **Upper Triangular Matrix**

Another **Direct Method**: for solving Ax=b: LU Decomposition

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1 & 2/3 \\ 1/3 & 2/3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 1/3 & 1/\sqrt{3} & \sqrt{5}/3 \end{bmatrix}\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & \sqrt{3}/2 & 1/\sqrt{3} \\ 0 & 0 & \sqrt{5}/3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Then, we can solve the linear system step-by-step.

Step 1: Define and solve for y

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 1/3 & 1/\sqrt{3} & \sqrt{5}/3 \end{bmatrix}}_{\underset{\sim}{L}}\underbrace{\underbrace{\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & \sqrt{3}/2 & 1/\sqrt{3} \\ 0 & 0 & \sqrt{5}/3 \end{bmatrix}}_{\underset{\sim}{U}}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}}}_{\vec{y}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{b}$$

$$\underset{\sim}{L} \cdot \vec{y} = \vec{b}$$

Solve for y: (forward substitution)

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 1/3 & 1/\sqrt{3} & \sqrt{5}/3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

easy to solve.

get: $y_1 = -1 \quad y_2 = \frac{5}{\sqrt{3}} \quad y_3 = \sqrt{5}$

Step 2: Solve for x

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & \sqrt{3}/2 & 1/\sqrt{3} \\ 0 & 0 & \sqrt{5}/3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

now that you know $\vec{y}$:

$$\underbrace{\underset{\sim}{U}}_{\text{known}} \cdot \vec{x} = \underbrace{\vec{y}}_{\text{known}}$$

Solve for x: (backward substitution)

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & \sqrt{3}/2 & 1/\sqrt{3} \\ 0 & 0 & \sqrt{5}/3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5/\sqrt{3} \\ \sqrt{5} \end{bmatrix}$$

easy to solve.

$x_3 = 3$. $x_2 = 4/3$, $x_1 = -8/3$

Summary:
How to solve Ax=b when you know L and U

$$A\vec{x} = \vec{b}$$

find L, U such that $L \cdot U = A$

$$LU = A \implies L\underbrace{U\vec{x}}_{} = \vec{b}$$

$$\text{Let } \vec{y} \equiv U\vec{x} \implies L\vec{y} = \vec{b},\ \text{solve for } y$$

$$U\vec{x} = \vec{y},\ \text{solve for } x$$

How do we find L and U? Approach 1

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

known unknown unknown

12 unknowns and not enough equations.

Look for a lower triangular matrix L and an upper triangular matrix U such that $A = L \cdot U$

How do we find L and U? Doolittle's Method

- The general problem of finding a lower triangular matrix and an upper triangular matrix whose product equals A does not have a unique solution. There can be many pairs (L, U) that multiply to make A.

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

- **Doolittle's Method** restricts the options available and gives a unique solution to the above problem.
  - Look for L and U of the form:

$$L = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

Given an A, find L and U such that $L \cdot U = A$

LU Decomposition using Doolittle's Method

- **Doolittle's Method** restricts the options available and gives a unique solution to the above problem.
  - Look for L and U of the form:
  - $$L = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$
  - Then LU is:

we want this to be equal to A.

$$L \cdot U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{bmatrix}$$

  - Perform Gaussian Elimination on this.

e.g. one step could be:

row 2 = row 2 − $(\overset{L_{21}}{\cdots})$ x row 1

what should these be?

row 3 = row 3 − $(\overset{L_{31}}{\cdots})$ x row 1

row 3 = row 3 − $(\overset{L_{32}}{\cdots})$ x row 2

What matrix are we left with?

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\begin{bmatrix} 9 & 5 & 6 \\ 5 & 8 & 5 \\ 1 & 2 & 1 \end{bmatrix}$$

→ Gauss-eliminate.
→ keep track of multipliers

Multipliers: $\left\{ \frac{5}{9}, \frac{1}{9}, \frac{13}{9} \times \frac{9}{47} \right\}$

Ans.

$$\begin{bmatrix} 9 & 5 & 6 \\ 0 & 47/9 & 5/3 \\ 0 & 0 & -6/47 \end{bmatrix}$$

Multiplier
row 1
row 2

row 2 = row 2 − ($5/9$) row 1 : $\left[5 - \frac{5}{9} \cdot 9,\ 8 - \frac{5}{9} \cdot 5,\ 5 - \frac{5}{9} \cdot 6\right]$

row 3 = row 3 − ($1/9$) row 1 $\left[1 - \frac{1}{9} \cdot 9,\ 2 - \frac{1}{9} \cdot 5,\ 1 - \frac{1}{9} \cdot 6\right]$

at this stage, our matrix is

$$\begin{bmatrix} 9 & 5 & 6 \\ 0 & 8 - \frac{25}{9} & 5 - \frac{30}{9} \\ 0 & 2 - 5/9 & 1 - 6/9 \end{bmatrix} = \begin{bmatrix} 9 & 5 & 6 \\ 0 & 47/9 & 15/9 \\ 0 & 13/9 & 3/9 \end{bmatrix}$$

now, do:

row 3 = row 3 − (?) row 2 → choose $\frac{13/9}{47/9}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 5/9 & 1 & 0 \\ 1/9 & 13/47 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 9 & 5 & 6 \\ 0 & 47/9 & 5/3 \\ 0 & 0 & -6/47 \end{bmatrix}$$

$$\underbrace{L \cdot U}_{A} \cdot \vec{x} = \vec{b}$$

call it $\vec{y}$. Solve $L \cdot \vec{y} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 5/9 & 1 & 0 \\ 1/9 & 13/47 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 8 \end{bmatrix}$$

Then find $\vec{x}$ from

$$U \cdot \vec{x} = \vec{y} \qquad \begin{bmatrix} 9 & 5 & 6 \\ 0 & 47/9 & 5/3 \\ 0 & 0 & -6/47 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

The same L and U can be used to solve $A\vec{x} = \vec{c}$ (page 4)

LU Decomposition using Doolittle's Method

- **Doolittle's Method** restricts the options available and gives a unique solution to the above problem.
  - The U of Doolittle's method is identical to the result of Gauss-eliminating the original matrix A
  - What about L ?
    - The off-diagonal elements of L are the values of the multipliers that were used during Gauss elimination

$$\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}$$

The multiplier that eliminated row 2, column 1 of our original Matrix when we Gauss-eliminated it.