

The Lorenz equations

$\sigma, r, b > 0$

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz\end{aligned}$$

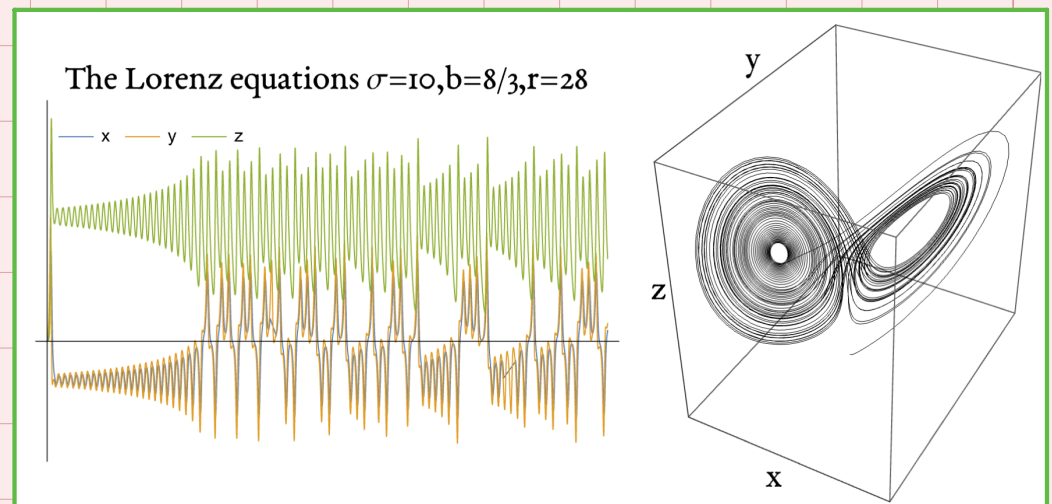
Simplified model of
atmospheric convection.

→ Nonlinear

→ 3-dimensional phase space

→ Symmetric : $(x, y, z) \mapsto (-x, -y, z)$

tinyurl.com/E91lorenz1
tinyurl.com/E91lorenz2



Wed, Apr 2 Lecture 18

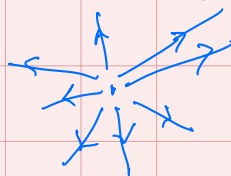
→ Dissipative : "Volumes in phase space" shrink exponentially with time

i.e. $\dot{\underline{x}} = \underline{f}(\underline{x})$, $\nabla \cdot \underline{f} < 0$ for dissipative systems.

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Because of dissipation

- Quasiperiodicity is not allowed because q.p. flow occurs on the surface of a fixed torus in phase space. But if volumes in phase space are always shrinking, you can't have such an invariant torus.
- No repelling fixed points or repelling closed orbits are allowed.



Fixed Points of Lorenz system

$(x, y, z) = \underline{0}$ is always a fixed pt.

$(x, y, z) = (\pm \sqrt{b(r-1)}, \pm \sqrt{b(r-1)}, r-1)$ are fixed pts if $r > 1$

(c^+, c^-)

Linear stability of origin:

linearized Lorenz equations:

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - y$$

$$\dot{z} = -bz$$

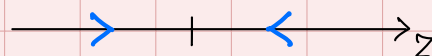
contracting direction

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma \\ r & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

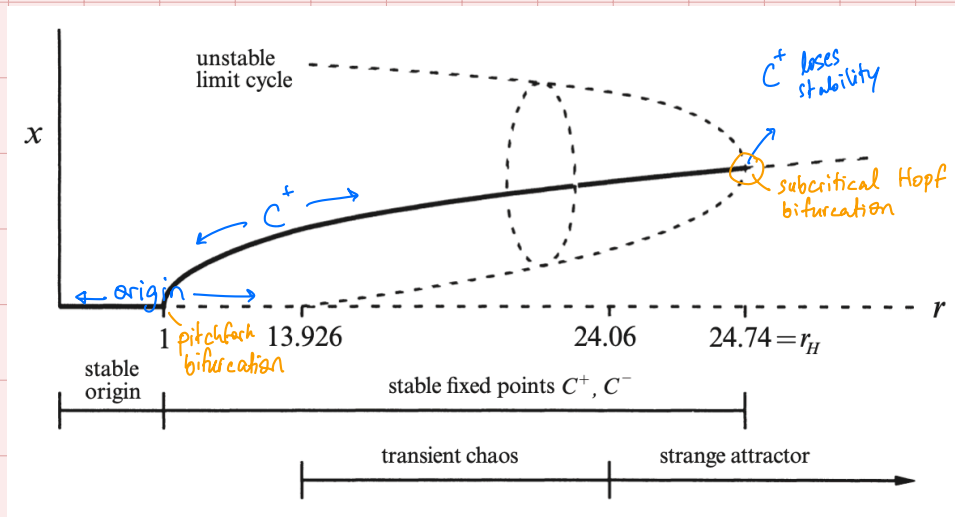
$$\tau = -(\sigma + 1)$$

$$\Delta = \sigma - \sigma r = \sigma(1 - r)$$

$\left. \begin{array}{l} \tau = -(\sigma + 1) \\ \Delta = \sigma - \sigma r = \sigma(1 - r) \end{array} \right\} \begin{array}{l} r > 1 : \text{saddle pt} \\ r < 1 : \text{stable node} \end{array}$



C^+ and C^- are stable for $1 < r < r_H = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$



so what exists after $r = r_H$?

- trajectories don't go out to infinity
- (it can be shown) there are no stable limit cycles
- no attracting fixed pts.
- but phase space is dissipative

<E91 Lorenz3>

→ A strange attractor

Attractor :

if "A" is an attractor, it has these properties →

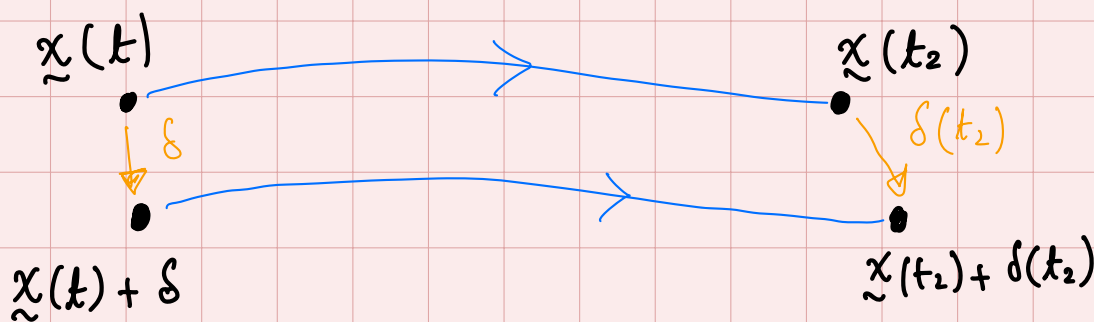
1. A is an *invariant set*: any trajectory $\mathbf{x}(t)$ that starts in A stays in A for all time.
2. A *attracts an open set of initial conditions*: there is an open set U containing A such that if $\mathbf{x}(0) \in U$, then the distance from $\mathbf{x}(t)$ to A tends to zero as $t \rightarrow \infty$. This means that A attracts all trajectories that start sufficiently close to it. The largest such U is called the *basin of attraction* of A .
3. A is *minimal*: there is no proper subset of A that satisfies conditions 1 and 2.

An attractor has a certain shape — what is the shape of the Lorenz attractor? — dimension "2.05"

tinyurl.com/E91lorenz3

For Lorenz's parameters $\sigma=10$, $b=8/3$, $r=28$, the system exhibits chaos. Strange attractors are chaotic ones.

Strogatz: Chaos is aperiodic long-term behaviour in a deterministic system that shows sensitive dependence on initial conditions.



$\delta(t)$:
how far apart are the two trajectories

use both as independent as initial conditions. \longrightarrow See how δ evolves

