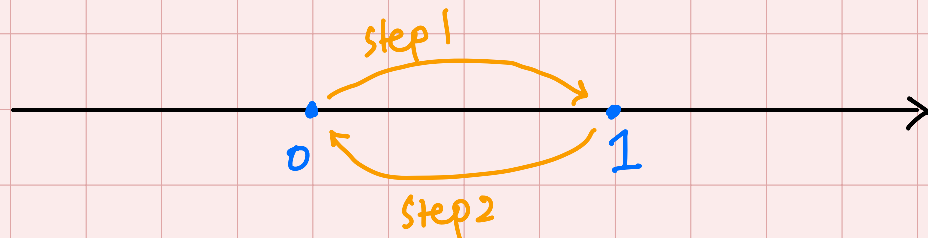
Mon, Apr 14 Lecture 21

Maps can have fixed points and periodic orb.

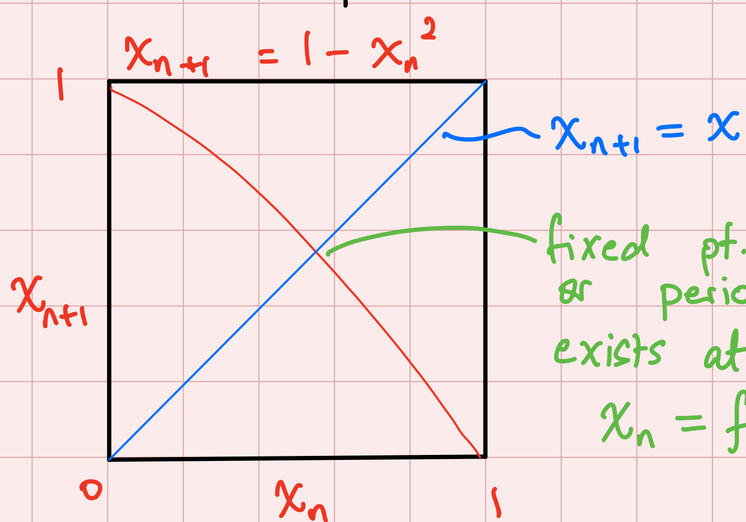
e.g. consider the map $x_{n+1} = 1 - x_n$ (*)

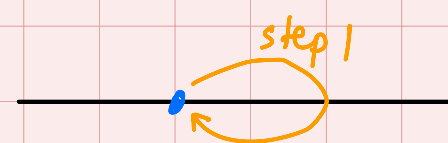
if $x_0 = 1$, $x_1 = 0$. $x_2 = 1$, $x_3 = 0$, ...



We have found a period − 2 orbit of the map (*)

Note: a fixed point can be called a period - 1 orbit.





fixed pt.
or period - 1 orbit.
exists at solution to eqn:

$x_n = f(x_n)$, or $x = 1 - x^2$

For a given f, how do we find period-n orbit ?

$x = f(f(x))$ for $n = 2$

Stability on Maps

If $x^* = f(x^*)$ for some map $x_{n+1} = f(x_n)$

Let's see what happens to $x = x^* + \eta$, for small η, under the action of this map.

When $\eta = 0$, solution remains at x.

$$x_{n+1} = f(x^* + \eta_n) = \underbrace{f(x^*)}_{x^*} + \eta_n f'(x^*) + \underbrace{\frac{\eta_n^2}{2!} f''(\dots) + \dots}_{\text{ignore}}$$

What happens to $x^* + \eta$ under map?

$$\boxed{\cancel{x^*} + \eta_{n+1} = \cancel{x^*} + f'(x^*)\,\eta_n}$$

$$\boxed{\eta_{n+1} = \underbrace{f'(x^*)}_{\text{call this } \lambda.}\,\eta_n}$$

for any given fixed pt. x^*, $f'(x^*)$ is just a number.

and get a linearized map.

$$\eta_{n+1} = \lambda \eta_n$$

$$\boxed{\eta_n = \lambda^n \eta_0}$$

$\eta_1 = \lambda \eta_0$

$\eta_2 = \lambda \eta_1 = \lambda^2 \eta_0$

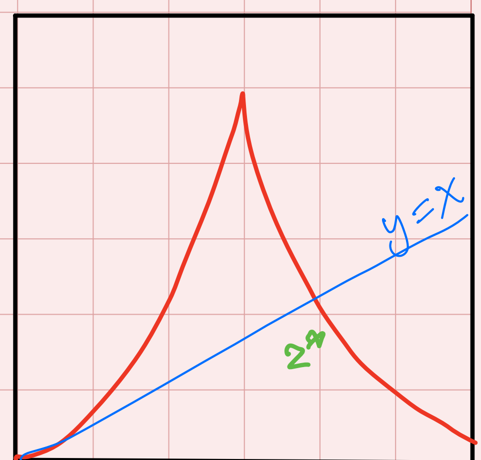
$\eta_3 = \lambda \eta_2 = \lambda^2 \eta_1 = \lambda^3 \eta_0$

if $|\lambda| < 1$, $\eta_n \to 0$ as $n \to \infty$

if $|\lambda| > 1$, $\eta_n \to \infty$ as $n \to \infty$

Evaluate Stability of fixed points in the Mathematical Model of the Lorenz map.

$$z_{n+1} = 1 - |2z_n - 1|^{1/2}$$



- where is the fixed pt?
- is it stable?

$$x = 1 - |2x - 1|^{1/2}$$

$$\sqrt{|2x-1|} = 1 - x$$

$$2x - 1 = (1-x)^2$$

$$\Rightarrow \quad x^2 - 4x + 2 = 0$$

$$x = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \frac{\sqrt{8}}{2} = 2 \pm \sqrt{2}. \qquad \boxed{z^* = 2 - \sqrt{2}}$$

$f'(x) = ?$ $\quad f(x) = 1 - \sqrt{2x-1}$, ignore the $|\cdot|$

$$f'(x) = -\frac{1}{2}(2x-1)^{-1/2} \cdot 2$$

$$= -\frac{1}{\sqrt{2x-1}}$$

Now, evaluate at $x = x^*$

$$f'(x^*) = -\frac{1}{\sqrt{4-2\sqrt{2}-1}} = -\frac{1}{\sqrt{3-2\sqrt{2}}} = -1-\sqrt{2} \approx -2.41$$

$$f'(x^*) \approx -2.41 \implies \text{unstable.}$$

The Logistic Map $\longrightarrow$ $x_{n+1} = r\, x_n (1 - x_n)$

$r \hookrightarrow$ parameter.

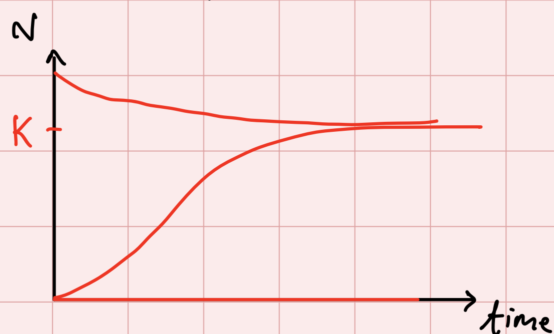
$0 < r \leq 4$

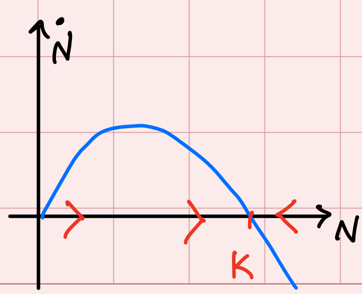
recall: $\dot{N} = rN\left[1 - \frac{N}{K}\right]$

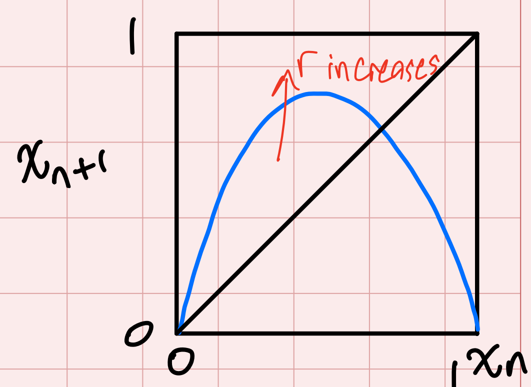
carrying capacity K

growth rate r

This system has the following behaviour:

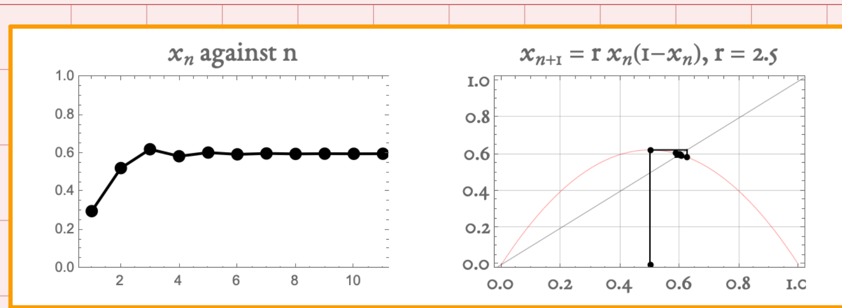




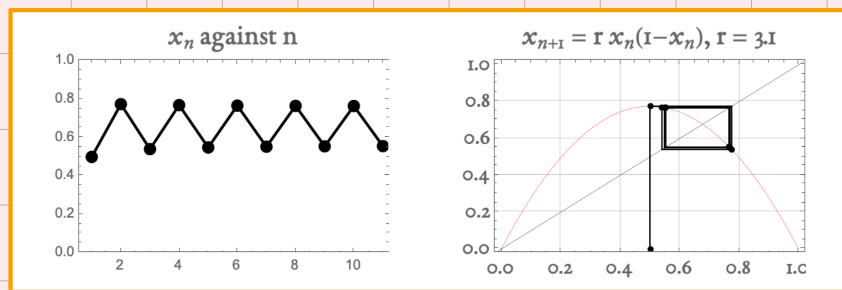


We now observe the logistic map's behaviour for different values of r

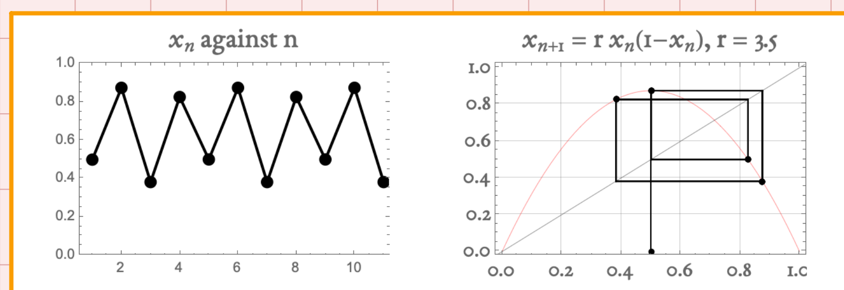
$r = 2.5$



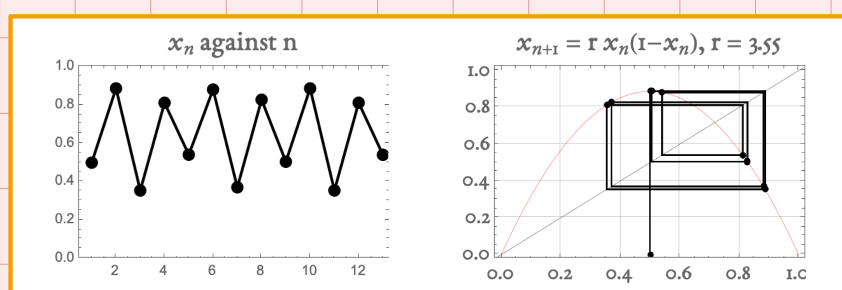
$r = 3.1$



$r = 3.5$



$r = 3.55$



$r = 3.565$