Wed, Jan 22 Lecture 1

Definition.
Determinism is true of the *world* if and only if, given a specified *way things are at a time t*, the way things go *thereafter* is *fixed* as a matter of *natural law*.

(Stanford Encyclopedia of Philosophy, Entry on Causal Determinism)

Laplace's Demon.
"We ought to regard the present state of the universe as the effect of its antecedent state and as the cause of the state that is to follow. An intelligence knowing all the forces acting in nature at a given instant, as well as the momentary positions of all things in the universe, would be able to comprehend in one single formula the motions of the largest bodies as well as the lightest atoms in the world, provided that its intellect were sufficiently powerful to subject all data to analysis; to it nothing would be uncertain, the future as well as the past would be present to its eyes. The perfection that the human mind has been able to give to astronomy affords but a feeble outline of such an intelligence. Discoveries in mechanics and geometry, coupled with those in universal gravitation, have brought the mind within reach of comprehending in the same analytical formula the past and the future state of the system of the world. All of the mind's efforts in the search for truth tend to approximate the intelligence we have just imagined, although it will forever remain infinitely remote from such an intelligence."

(1820)

(Essai Philosophique sur les Probabilités)

Principle of Sufficient Reason - Leibniz

"Free Will is an illusion" - Spinoza

Heisenberg Uncertainty Principle

Gödel's Incompleteness Theorem

$$\left\{ \begin{aligned} \dot{x} &= f(x,t) \\ x(0) &= x_0 \end{aligned} \right\} \; (*)$$

$x \in \mathbb{R}^n$

Chaos is impossible if $n < 3$

To "solve" this IVP means to find a function $x(t)$ that satisfies $(*)$.

- analytical solution : use MATH
- numerical solution : use computer

A solution may exist for all $t \in \mathbb{R}$ or for a subset of $\mathbb{R}$

Mon, Jan 27 Lecture 2

$\dot{x} = f(x,t)$: first-order

$\ddot{x} = f(x,\dot{x},t)$: 2^{nd}

$\dddot{x} = f(x,\dot{x},\ddot{x},t)$: 3^{rd}

e.g. $x(0) = ..$

$\dot{x}(0) = ..$

$\ddot{x}(0) = ..$

$\ddot{x} = f(x,\dot{x})$: autonomous

$\ddot{x} = f(x,\dot{x},t)$: nonautonomous

Equivalence of n^{th} order differential equations and a system of n 1^{st} order " "

$$\frac{d^n x}{dt^n} = f\left(x, \dot{x}, \ddot{x}, x^{(3)}, x^{(4)}, \ldots, x^{(n-1)}\right)$$

it is always possible to write an equivalent system of n 1^{st} order equations:

Define $y \in \mathbb{R}^n$

$$\begin{aligned}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= y_3 \\
\dot{y}_3 &= y_4 \\
&\vdots \\
\dot{y}_{n-1} &= y_n \\
\dot{y}_n &= f(y_1, y_2, y_3, \ldots)
\end{aligned}$$

Any dynamics problem can be written as

$$\dot{\vec{x}} = f(\vec{x}, t) \quad \text{where} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \dot{\vec{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$

$\dot{x} = f(x)$, $x \in \mathbb{R}^n$
$x(0) = x_0$

is linear if it can be written as $\dot{x} = A x$ (A: $n \times n$)

$n = 1$, linear → increasing n

↓

increasing nonlinearity

chaos lives here

$n \gg 1$, nonlinear

$n = 1$, nonlinear

$\dot{x} = \sin x$
$x(0) = x_0$

→ analytical : $x(t)$ function
numerical : $\{t_i, x_i\}$
geometric

① Analytical

$$\frac{dx}{dt} = \sin x$$

$$\int \operatorname{cosec} x \, dx = \int dt$$

use $t = 0$, $x = x_0$ to find C

$$-\log|\operatorname{cosec} x + \cot x| + C = t$$

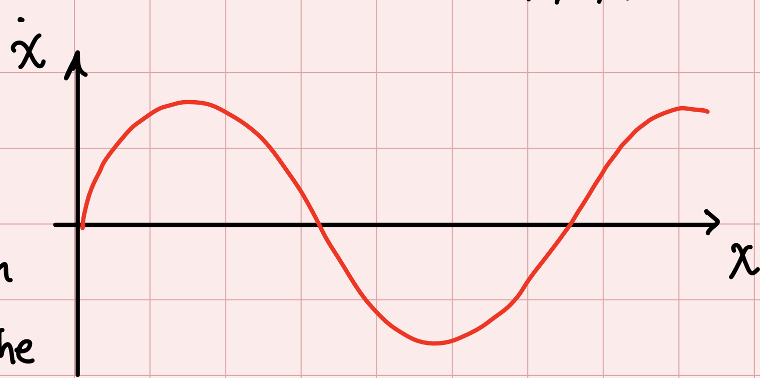
$$\log\left|\frac{\operatorname{cosec} x_0 + \cot x_0}{\operatorname{cosec} x + \cot x}\right| = t$$

② **Numerical** $\quad \frac{dx}{dt} = \sin x$

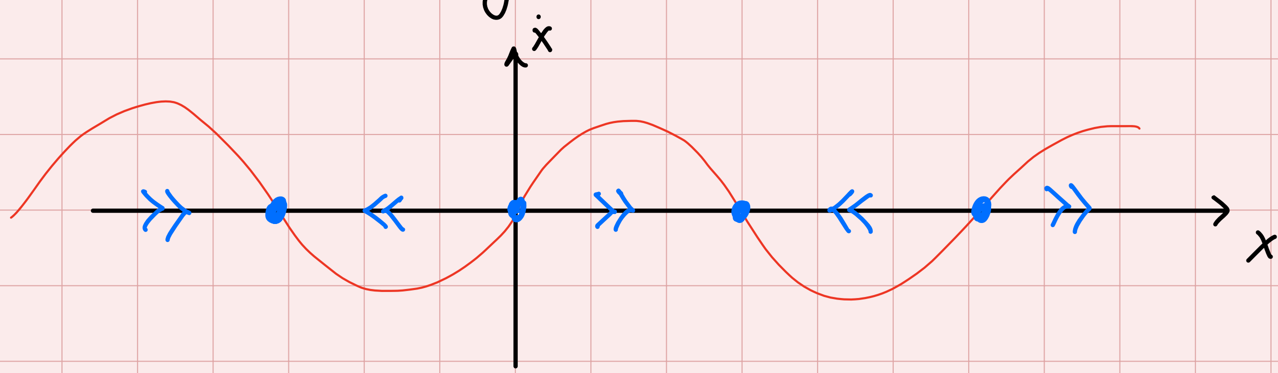
$$\frac{x_{n+1} - x_n}{\Delta t} = \sin x_n \Rightarrow x_{n+1} = \Delta t \sin(x_n) + x_n$$

$n = 0, 1, 2, 3, \ldots$

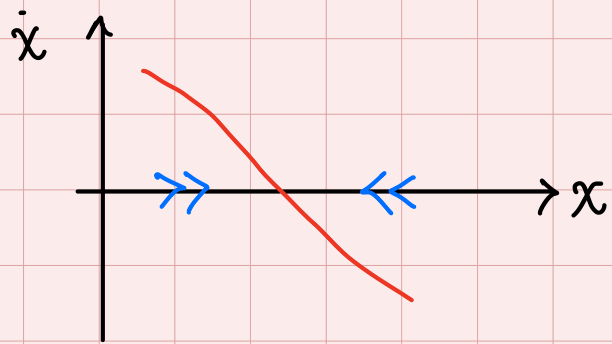
③ **Geometric**



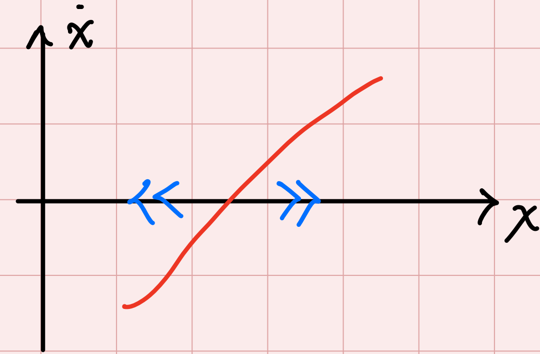
- The state of system is a point on the x-axis
- $\dot{x} > 0$: moving to the right
  $\dot{x} < 0$: moving to the left. } call this "flow"



Two kinds of "fixed points" emerge:

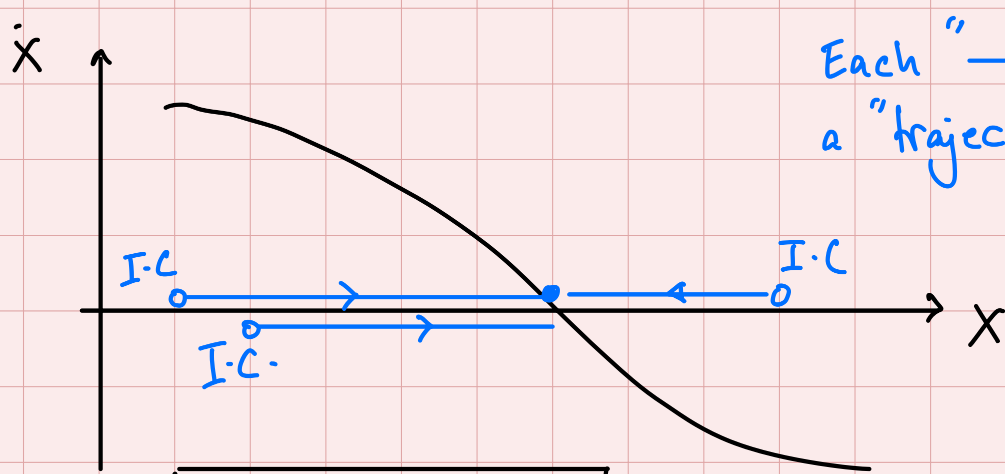


ATTRACTOR
SINK
(stable)



REPELLER
SOURCE
(unstable)

Phase Portrait (for 1-d systems)



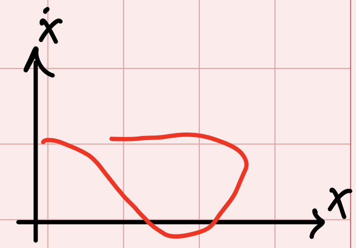
Each "→" is a "trajectory".

System $\dot{x} = f(x)$

- one phase portrait
- (a few?) fixed points
- infinite trajectories

A diagram showing
- all qualitatively different trajectories
- all fixed points

Note: $f(x)$ must be a function



not allowed

in addition, we will work with $[f(x)]$'s that are "nice"

= sufficiently smooth.

Wed, Jan 29 Lecture 3

Example $\dot{N} = rN\left[1 - \frac{N}{K}\right]$ Logistic Equation

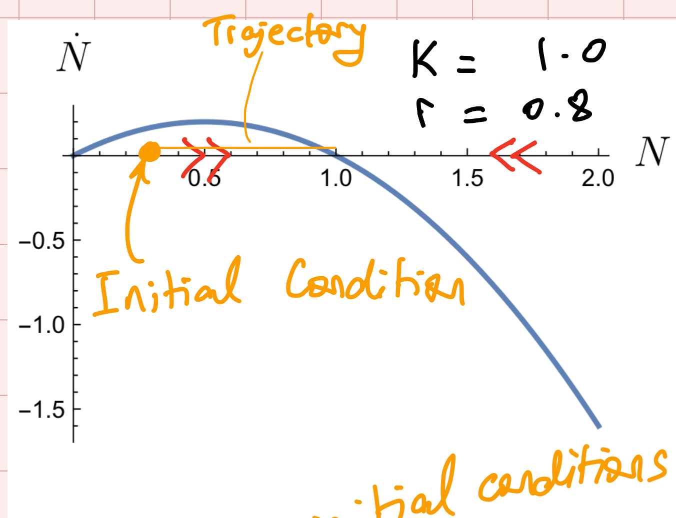
K → carrying capacity

$[N]$ = people

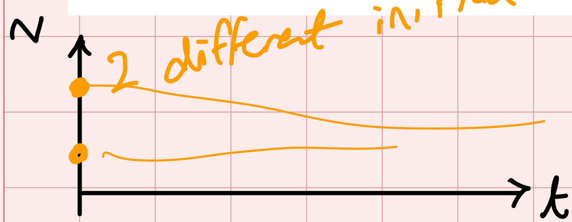
$[r]$ = day^{-1}

$[K]$ = people

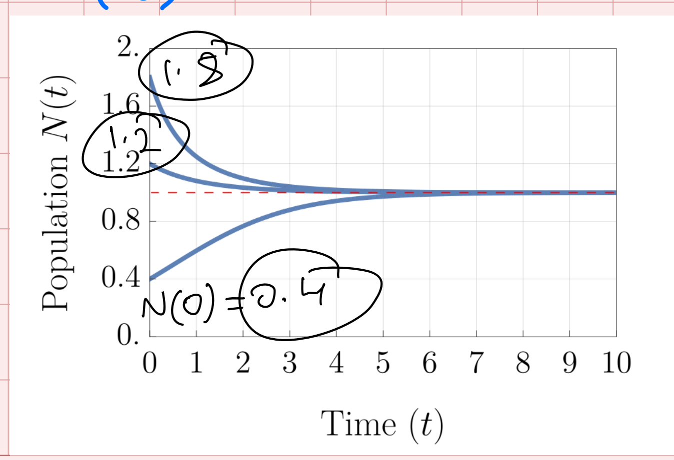
$[\dot{N}]$ = people/day



2 different initial conditions



$N(t)$

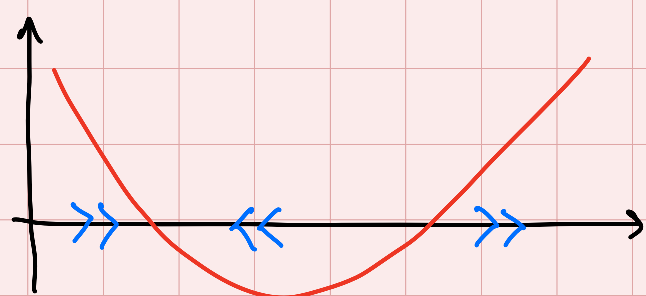


Calculating curves $N(t)$:

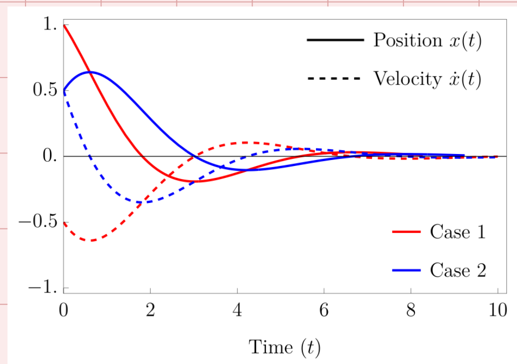
$$\frac{dx}{dt} = x(1-x) \quad \text{with } x(0) = x_0 \text{ or, } 1/2$$

<u>Note</u> : It can be shown that oscillations and "overshoot", or other <u>nonmonotonic behaviour</u> is impossible in

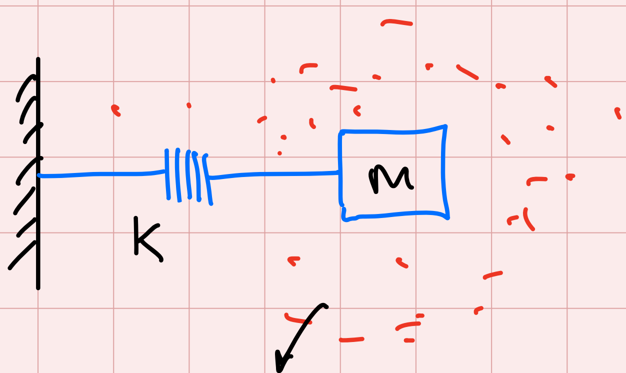
$$\dot{x} = f(x), \quad x \in \mathbb{R}^1$$



e.g.



in S.H.O.



very viscous fluid, c

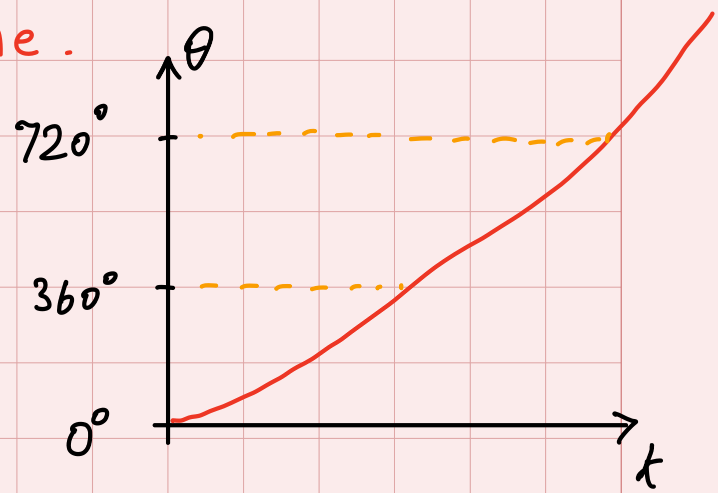
$$m\ddot{x} + c\dot{x} + kx = 0$$

drop ($m\ddot{x}$)

But : $\dot{\theta} = f(\theta), \quad \theta \in [0, 2\pi)$ can have "oscillations"

we reinterpret $\theta = 370^\circ$ to mean
$\theta = 10^\circ$.

i.e. θ lives on the circle here, not on the real line.



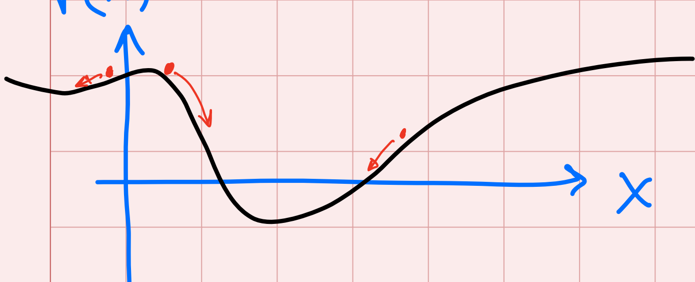
Potentials

if $\dot{x} = f(x)$ and $f(x)$ can be expressed as

$$f(x) = -\frac{d}{dx} V(x)$$

for some scalar function $V(x)$

then $V(x)$ is called a "potential" for the dynamical system $\dot{x} = f(x)$

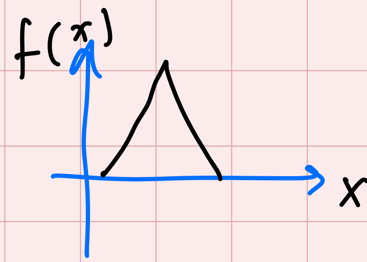


Flow occurs "downhill" in V.

Strogatz (2.7) shows that $\frac{dV}{dt} \le 0$ along trajectories $x(t)$.

$f(x) = 2x \longrightarrow V(x) = -x^2$

$f(x) =$



Linear Stability Analysis of fixed points.

Suppose x^* is a value of x where $f(x^*)=0$
What happens to x if it is initialized close to x^*?

Let $\eta(t) \equiv x(t) - x^*$

$\dot\eta(t) = \underbrace{\dot x(t)}_{f(x)} - 0$

$\dot\eta = \dot x = f(x)$

$\dot\eta = f(x) = f(x^* + \eta)$

use Taylor series assuming η small.

$f(x^*+\eta) = \cancelto{0}{f(x^*)} + \eta f'(x^*) + \underbrace{\frac{\eta^2}{2!} f''(x^*) + \cdots}_{O(\eta^2)}$

$$\boxed{\dot\eta = \eta \underbrace{f'(x^*)}_{\text{const. number}} + O(\eta^2)}$$

Evolution equation for small perturbations η away from x^*.

Mon, Feb 3 Lecture 4

Bifurcations

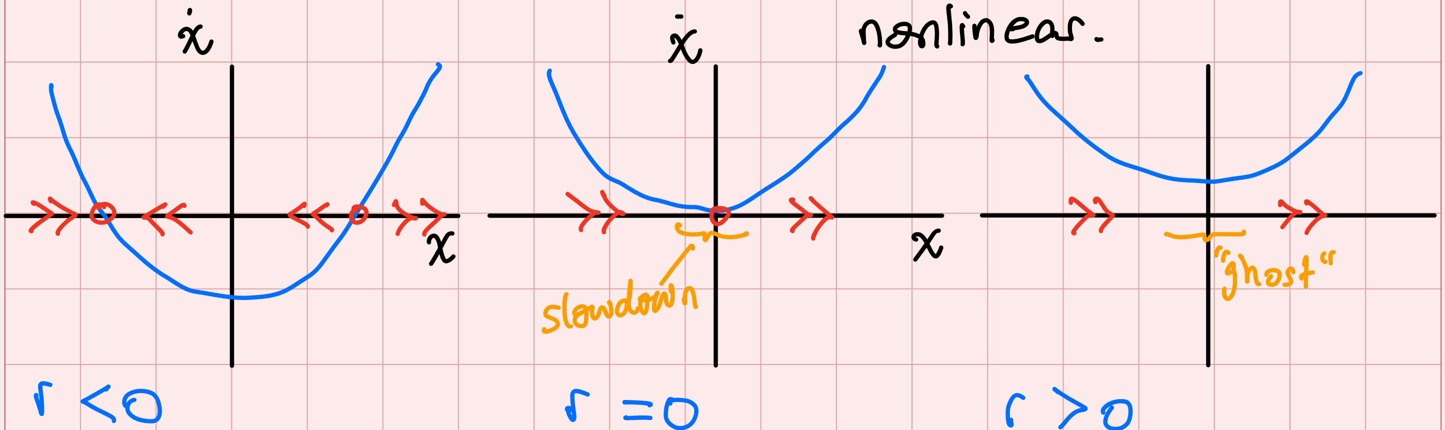
$$\dot{x} = f(\underbrace{x}_{\text{independent variable}};\ \underbrace{r}_{\text{parameter}})$$

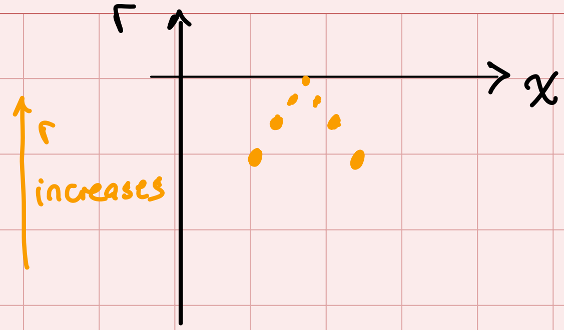
Let the system be parameterized by one or more parameters 'r'. How does the qualitative behaviour of the system change with r?

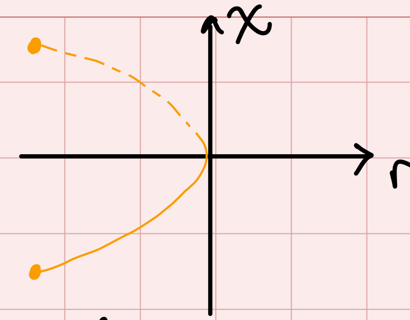
① Saddle-Node Bifurcation

$$\dot{x} = r + x^2$$

one-dimensional first-order nonlinear.



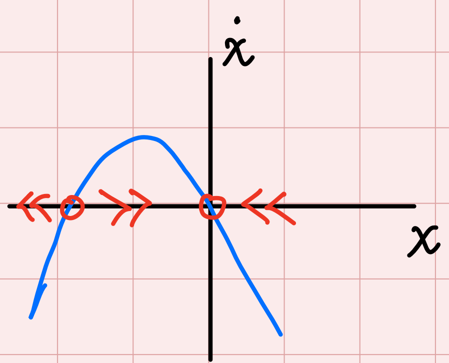




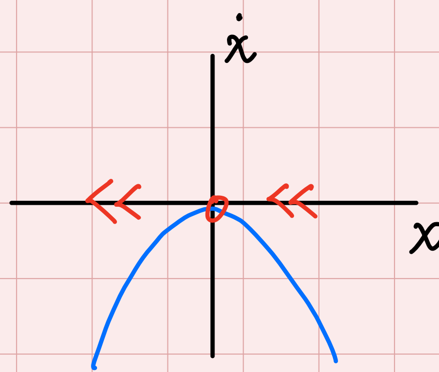
Bifurcation Diagram for Saddle-Node Bifurcation

② Transcritical Bifurcation

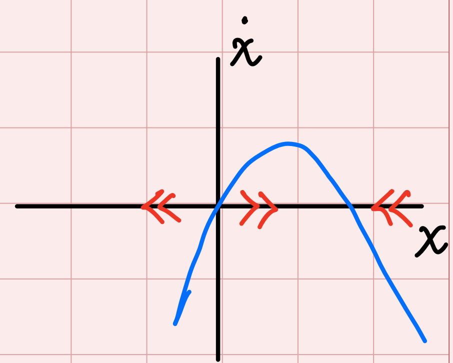
$$\dot{x} = rx - x^2$$



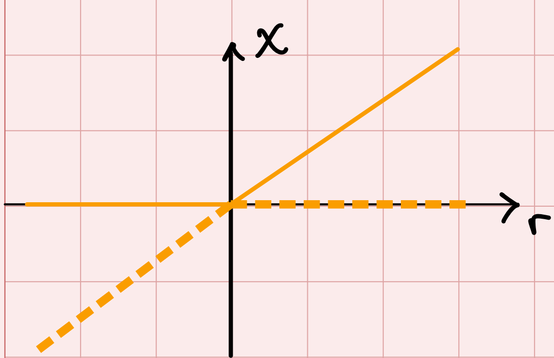
$r < 0$



$r = 0$



$r > 0$



"exchange of stability"

Bifurcation Diagram for Transcritical Bifurcation

How to calculate Bifurcation Curves

$\dot{x} = -x + r \tanh x$ (A)

Find fixed points x^*, for which $f(x^*) = 0$

Here, f also has a parameter r.

Solve $\boxed{f(x^*; r) = 0}$ for many r's.

Gives $\{x^*, r\}$ pairs. Plot them.

→ with root-finding program or by hand.

e.g. for system (A)

Solve $0 = -x^* + r \tanh x^*$ for x^*
after setting r to some value.

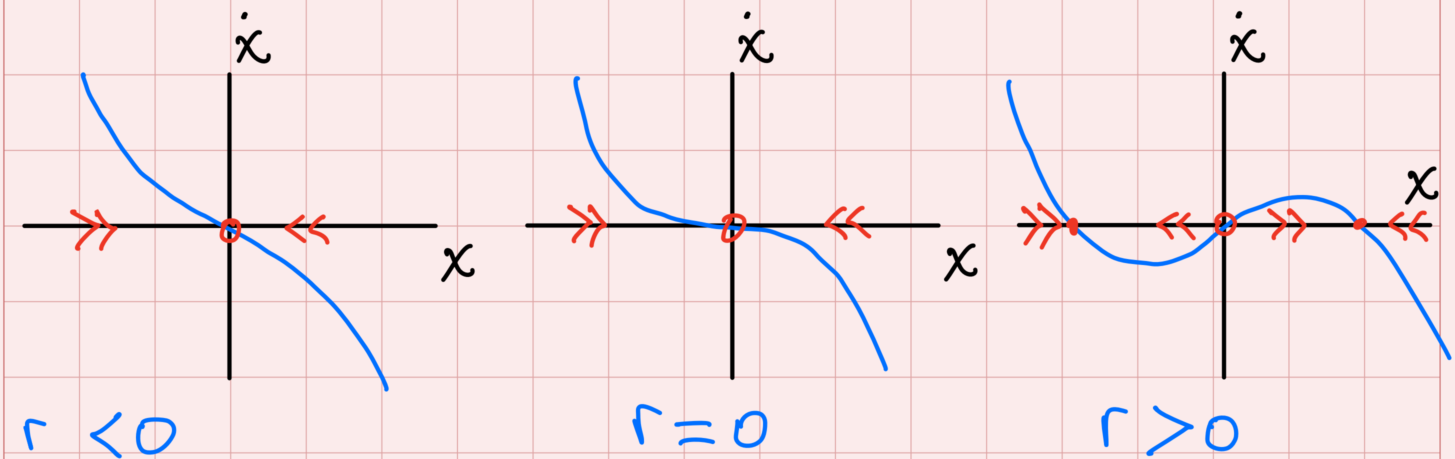
Set x^* to some value, find $r = \dfrac{x^*}{\tanh x^*}$

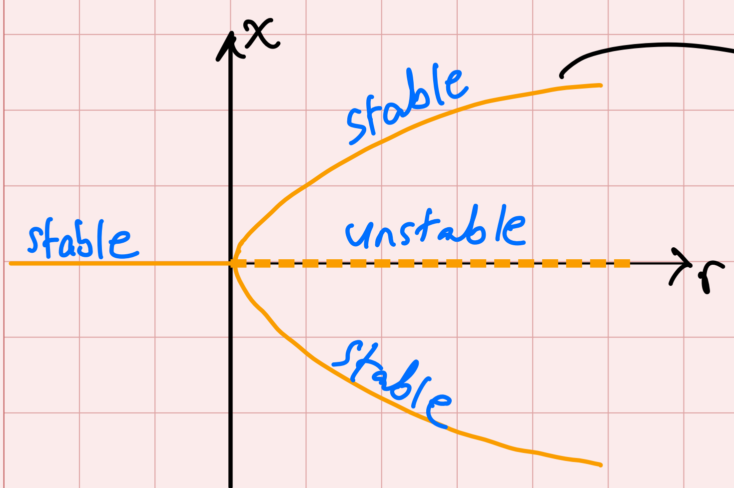
③ Pitchfork Bifurcation

$\dot{x} = rx - x^3$ supercritical

$\dot{x} = rx + x^3$ subcritical

→ supercritical : new fixed pts. appear <u>above</u> critical r.





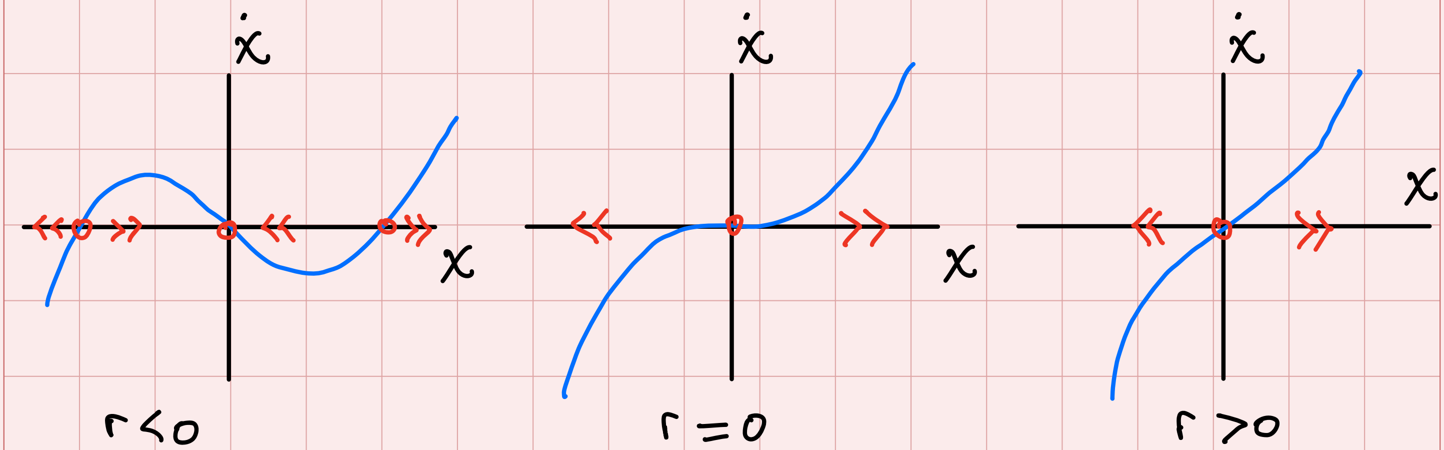
$x^*(r) = \dots ? \sqrt{r} \dots$

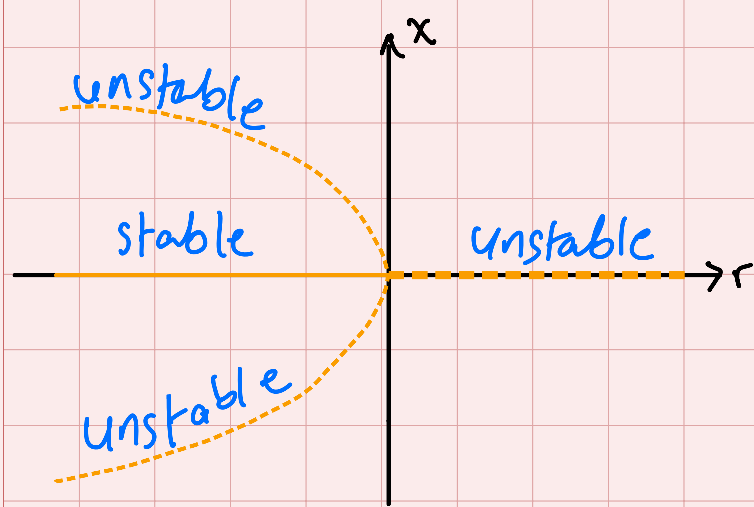
Bifurcation diagram:

$\dot{x} = 0$: solve for $\{r, x\}$

Wed, Feb 5 **Lecture 5**

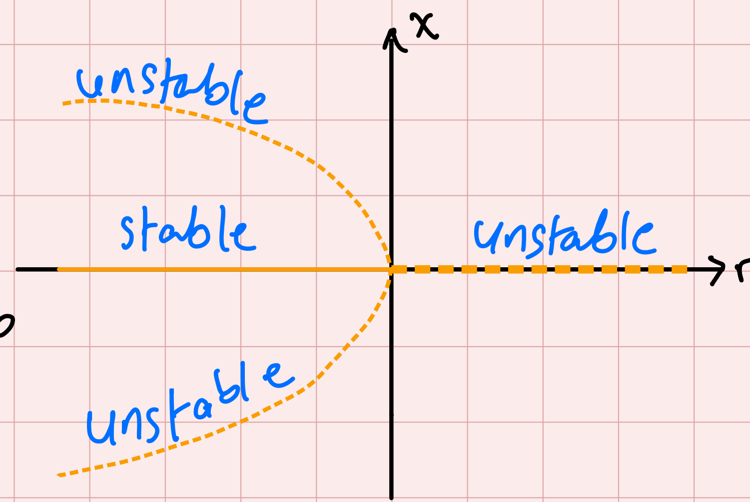
→ subcritical pitchfork $\dot{x} = rx + x^3$





	$\dot{x} = f(x;r)$	$r < 0$	$r > 0$
Saddle-node	$\dot{x} = r + x^2$	1 stable 1 unstable	None None
Transcritical	$\dot{x} = rx - x^2$	1 stable → 1 unstable →	1 unstable 1 stable
Pitchfork:			
Supercrit.	$\dot{x} = rx - x^3$	None → 1 stable → None →	1 stable 1 unstable 1 stable
subcrit.	$\dot{x} = rx + x^3$	1 unstable → 1 stable → 1 unstable →	None 1 unstable None

In practice, systems with a subcritical pitchfork bifurcation don't actually go off to $\pm\infty$ instead, higher-order terms play a stabilizing role.

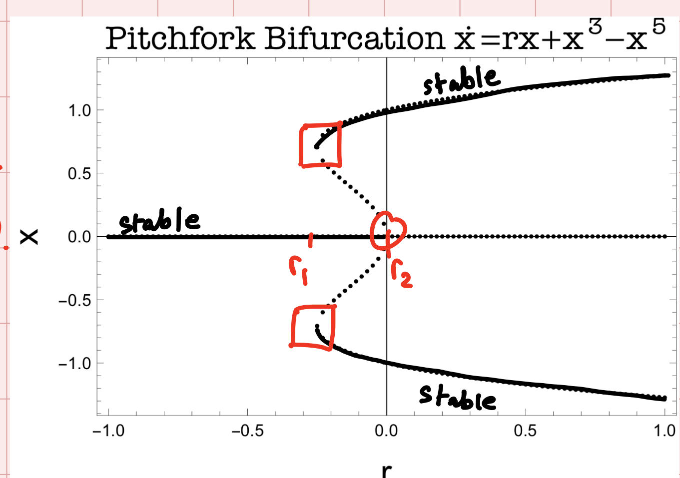


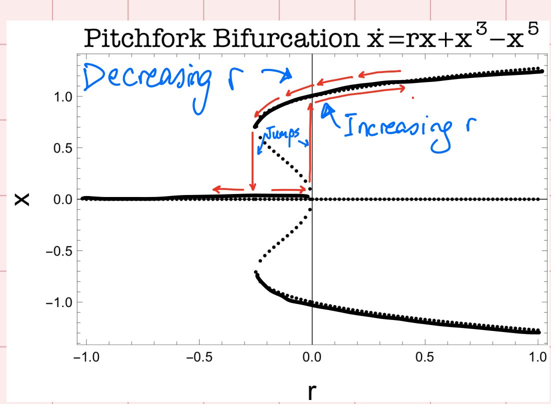
subcritical pitchfork

$$\underbrace{\dot{x} = rx + x^3}\, - x^5$$

for small x, this governs the dynamics.
for large x, the x^5 term plays a role also.

→ Saddle-node bifurcation at □.
→ Subcritical pitchfork bif. at ○.





Hysteresis

Mon, Feb 10 Lecture 6

Show that, with appropriate non-dimensionalization,

$\dot{u} = a u + b u^3 - c u^5$ is equivalent to

$\dot{x} = r x + x^3 - x^5$. where $\dot{x} = \frac{dx}{d\tau}$

$$U = \sqrt{b/c}$$
$$T = c/b^2$$
$$r = ac/b^2$$

$\leftarrow$ $\left\{ \begin{array}{l} x = u/U \\ \tau = t/T \end{array} \right.$

Dynamics with $n = 2$

$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}) \qquad \underset{\sim}{x} \in \mathbb{R}^2$$

<u>Conventions</u> the state can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ or } \begin{bmatrix} x \\ y \end{bmatrix} \text{ or } \underset{\sim}{x}$$

<u>Note</u>: f is a vector-valued function of a vector argument.

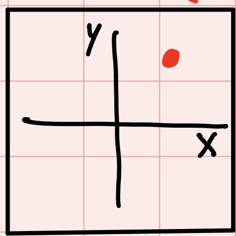
1-dimensional state

only one variable determines state

change: $\dot{x}$

2-dimensional state

(x, y) — determines state



change: $\dot{x}, \dot{y}$

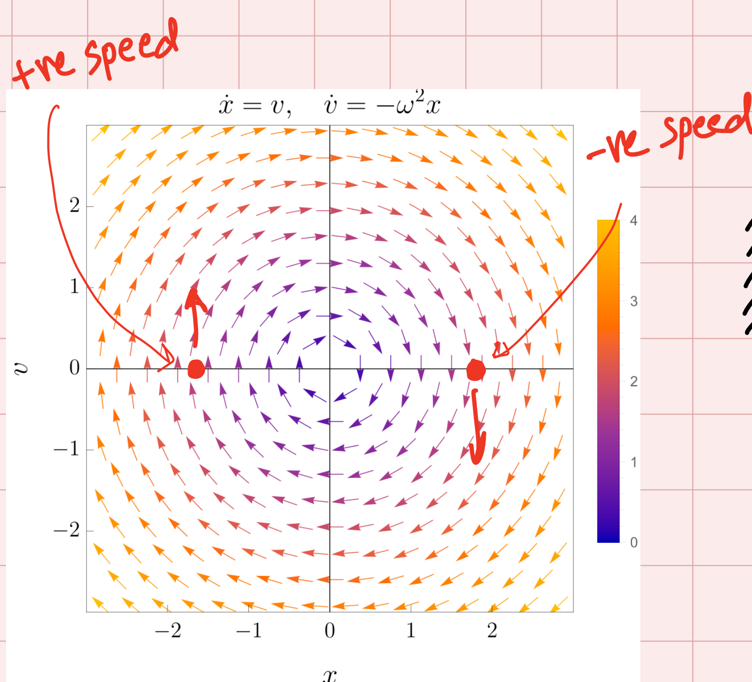
The solutions of $\left\{ \dot{\underset{\sim}{x}} = f(\underset{\sim}{x}),\ \underset{\sim}{x} \in \mathbb{R}^2,\ \underset{\sim}{x}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right\}$

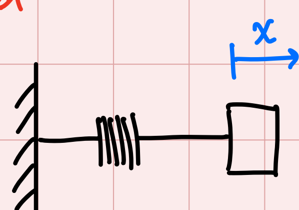
can be visualized as trajectories on the phase plane

$$\ddot{x} = -\omega^2 x \qquad \begin{matrix} \dot{x} \\ \ddot{x} \end{matrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

state: x, v — position, velocity

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ -\omega^2 x \end{bmatrix}$$





→ $\dot{\underset{\sim}{x}}$ is a vector with 2 components, defined for any point (x,y) on the plane.
i.e. $\dot{\underset{\sim}{x}}$ is a vector field

→ Trajectories are $\{x_1(t), x_2(t)\}$ functions of time parameterized by the initial condition.

or, numerically, ordered pairs parameterized by t.

→ Trajectories are everywhere tangent to vector field.

---

if $f(\underset{\sim}{x})$ is linear, without loss of generality we can express $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ as:

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note

$\underset{\sim}{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is always a fixed pt

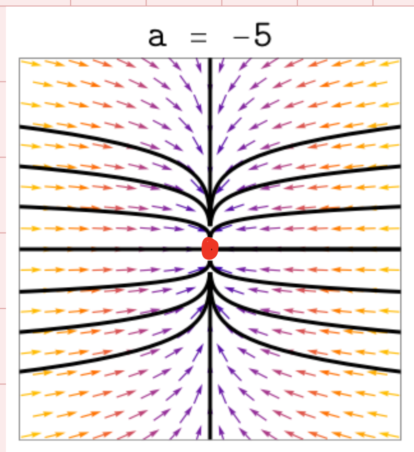
study a particular linear system:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

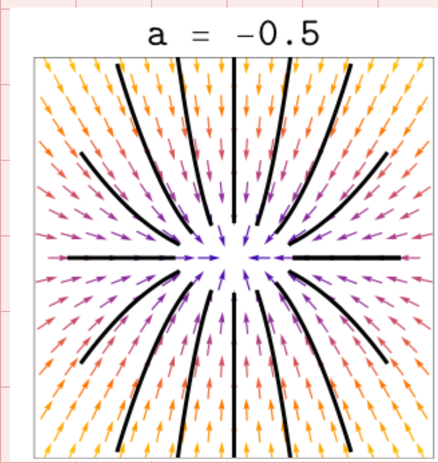
$\dot{x} = ax \longrightarrow x(t) = x_0 e^{at}$

$\dot{y} = -y \qquad y(t) = y_0 e^{-t}$

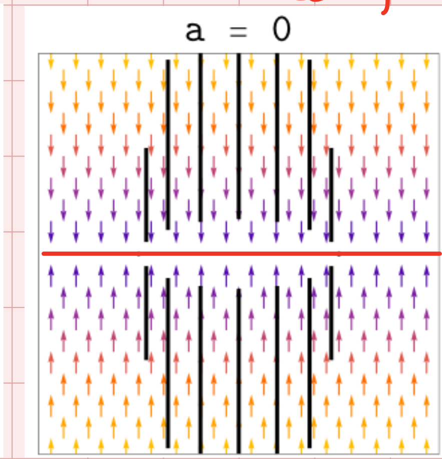
stable node

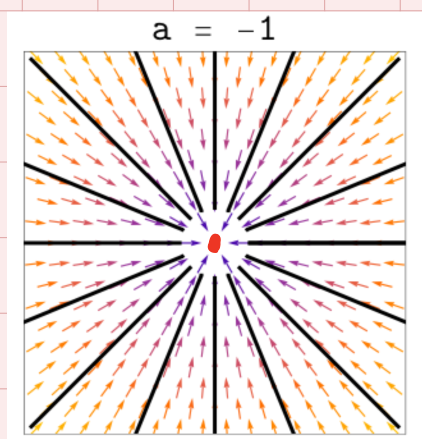


stable node

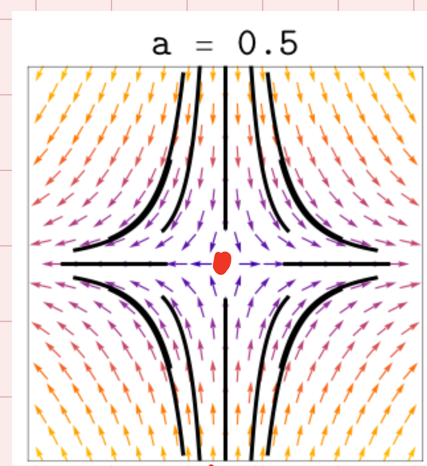


line of fixed pts





symmetric node (star)



saddle point

Wed, Feb 12, Lecture 7

General $n=2$ Linear Systems $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \overbrace{\begin{bmatrix}a & b\\ c & d\end{bmatrix}}^{A}\begin{bmatrix}x\\ y\end{bmatrix}$

Recall that, for $A=\begin{bmatrix}a & 0\\ 0 & -1\end{bmatrix}$, the x and y axes were special directions with straight-line trajectories

special directions → related to eigenvectors of A.

straight-line → exponential in time

Are there such trajectories for general A ?
What directions do those trajectories travel in ?
i.e. is there a vector $\underset{\sim}{v}$ and λ such that

$$\underset{\sim}{x}(t) = e^{\lambda t}\underset{\sim}{v} \quad ? \qquad \dot{\underset{\sim}{x}} = A\underset{\sim}{x}$$

$$\lambda e^{\lambda t}\underset{\sim}{v} = A e^{\lambda t}\underset{\sim}{v}$$

$$\lambda \underset{\sim}{v} = A\underset{\sim}{v}$$

: λ eigenvalues of A
$\underset{\sim}{v}$ eigenvectors of A.

$$\det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$

$$(a-\lambda)(d-\lambda) - cb = 0$$

$$\lambda^2 - \lambda a - \lambda d + ad - bc = 0$$

$$\lambda^2 - \underbrace{(a+d)}_{\mathrm{tr}(A),\ \tau}\lambda + \underbrace{ad - bc}_{\det(A),\ \Delta} = 0$$

tr: trace
det: determinant.

$$\boxed{\lambda^2 - \tau\lambda + \Delta = 0}$$

Equation for eigenvalues of A.

$$\boxed{\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}}$$

Note : once you know λ, it is straightforward to calculate $\underset{\sim}{v}$ by solving $\lambda\underset{\sim}{v} = A\underset{\sim}{v}$ for the two components of $\underset{\sim}{v}$.

As long as $\lambda_1 \neq \lambda_2$, any state of the system $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ can be written as a linear combination of the eigenvectors $\underset{\sim}{v_1}$ and $\underset{\sim}{v_2}$.

$$\underset{\sim}{x} = a_1 \underset{\sim}{v_1} + a_2 \underset{\sim}{v_2} \quad \text{for scalars } a_1 \text{ and } a_2.$$

$$\underset{\sim}{x}(t) = (a_1(t))\underset{\sim}{v_1} + (a_2(t))\underset{\sim}{v_2}$$

As $\underset{\sim}{x}$ varies over time, these scalars vary exponentially in time.

and it's possible to write a general solution $\underset{\sim}{x}(t)$ for the differential equation $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$.

$$\underset{\sim}{x}(t) = c_1 e^{\lambda_1 t} \underset{\sim}{v_1} + c_2 e^{\lambda_2 t} \underset{\sim}{v_2}$$

→ λ's, $\underset{\sim}{v}$'s are eigenvalues and eigenvectors.

→ c's are const. coefficients that depend on initial condition $\underset{\sim}{x}(0)$.

No such general solution exists for $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ if f is not linear in $\underset{\sim}{x}$.

Exercise Solve $\dot{x} = x + y \quad x_0 = 2$

$\dot{y} = 4x - 2y \quad y_0 = -3$

$\dot{x} = Ax$ with $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$. $\tau = -1$, $\Delta = -6$

First, find eigenvalues.

$$\lambda^2 + \lambda - 6 = 0 \Rightarrow \lambda^2 + 3\lambda - 2\lambda - 6 = 0$$

$$(\lambda + 3)(\lambda - 2) = 0$$

$$\Rightarrow \lambda = 2, -3$$

Then find eigenvectors.

$$\underset{\sim}{A}\,\underset{\sim}{u} = \lambda \underset{\sim}{u} \quad (\lambda = 2)$$

$$\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2u_1 \\ 2u_2 \end{bmatrix} \Rightarrow \begin{matrix} u_1 + u_2 = 2u_1 \\ 4u_1 - 2u_2 = 2u_2 \end{matrix} \Rightarrow \begin{matrix} u_1 = 1 \\ u_2 = 1 \end{matrix}$$

$$\Rightarrow \underset{\sim}{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

one eigenvector, associated with $\lambda = 2$

similarly .... $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$ is the 2nd eigenvector associated with $\lambda = -3$

$$\underset{\sim}{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

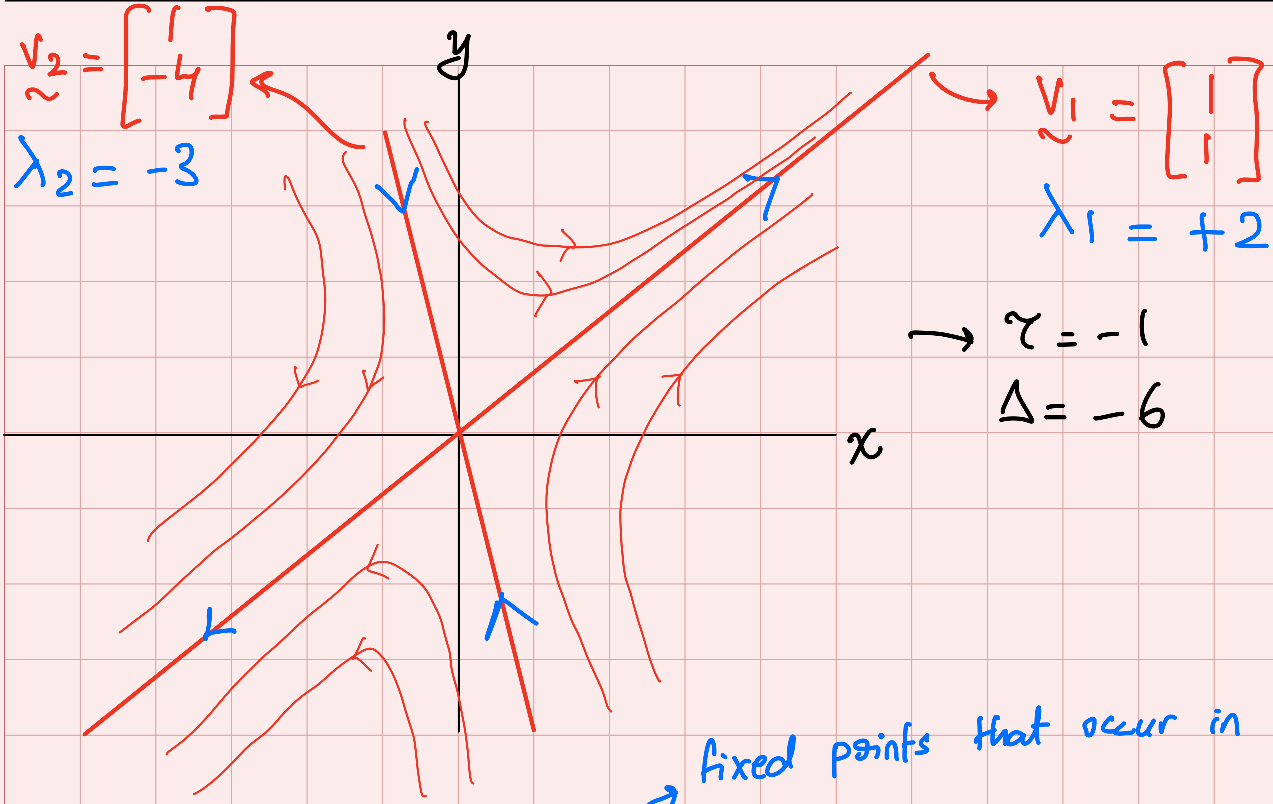
at $t=0$, $\underset{\sim}{x} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

at $t=0$

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} = c_1 e^0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^0 \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$c_1 + c_2 = 2$$
$$c_1 - 4c_2 = -3$$
$$\Rightarrow c_1 = 1, \quad c_2 = 1$$

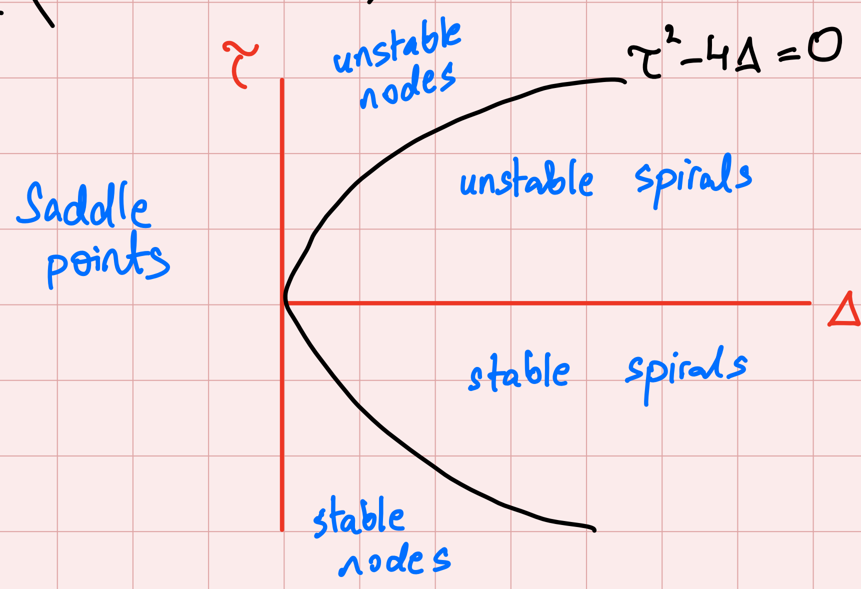
$$\underset{\sim}{x}(t) = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

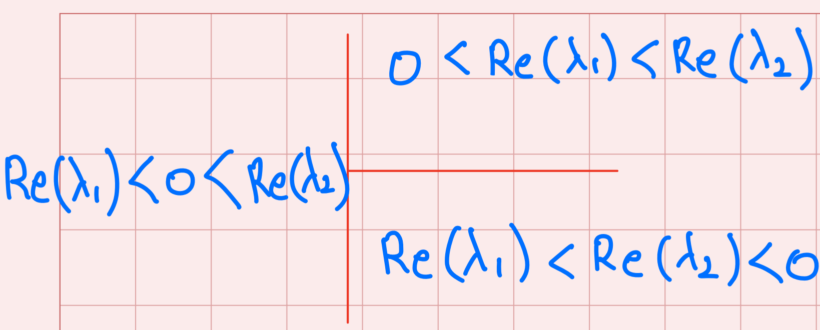


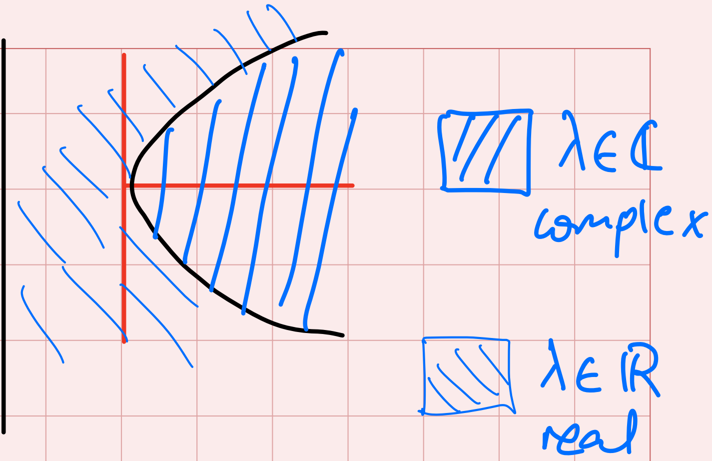
fixed points that occur in

A classification of any $n=2$ linear system

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$




0 < Re(λ1) < Re(λ2)
Re(λ1) < 0 < Re(λ2)
Re(λ1) < Re(λ2) < 0


λ ∈ ℂ
complex
λ ∈ ℝ
real

Mon, Feb 17 Lecture 8

Romeo & Juliet

constants

$R(t)$: $\dot{R} = \_\_\_ R + \_\_\_ J + ?$

$J(t)$: $\dot{J} = \_\_\_ R + \_\_\_ J + ?$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{constants}} \begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{\begin{bmatrix} \; \\ \; \end{bmatrix}}_{\text{constants.}}$$

+ve : love
−ve : hate

R : Romeo's love / hate for Juliet
J : Juliet's love / hate for Romeo

rate of change of R depends on value of R
" " J

NOT on rate of change of R, J.

"Romantic styles"

$\dot{x} = +x + y$

$\dot{x} = +x - y$

$\dot{x} = -x + y$

$\dot{x} = -x - y$

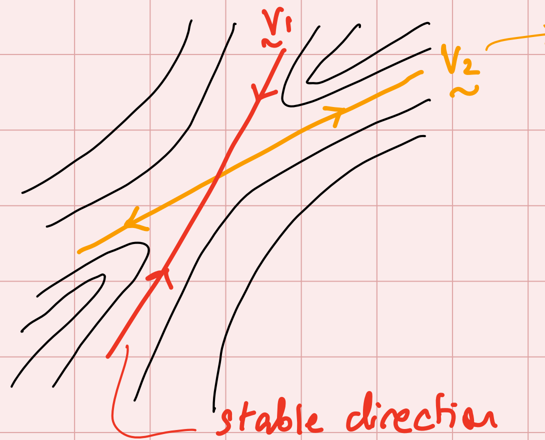
x : one lover
y : other one

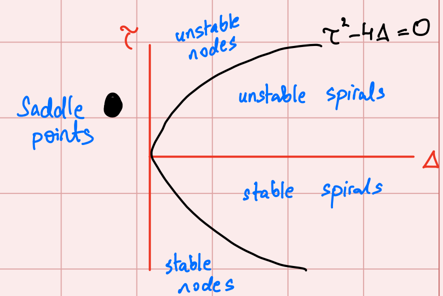
Wed, Feb 19 Lecture 9

Fixed Point Types and eigenvalues/eigenvectors

① Saddle Point

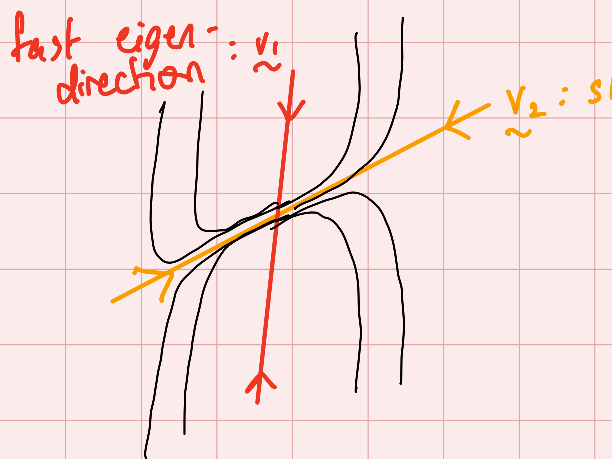
$$\underbrace{\lambda_1}_{\underset{\sim}{v_1}} < 0 < \underbrace{\lambda_2}_{\underset{\sim}{v_2}}$$

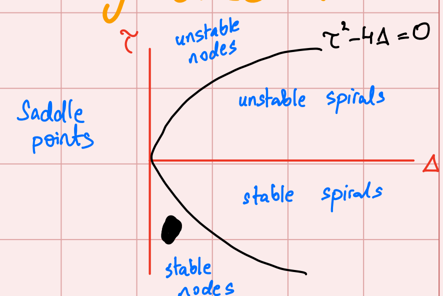




② Nodes

$$\underbrace{\lambda_1}_{\underset{\sim}{v_1}} < \underbrace{\lambda_2}_{\underset{\sim}{v_2}} < 0$$

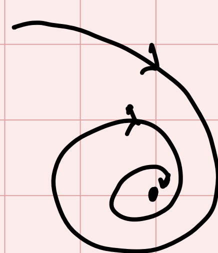


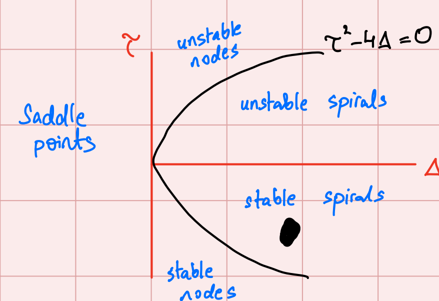


③ Spirals

$\mathrm{Re}(\lambda)$: exponential decay

$\mathrm{Im}(\lambda)$: oscillation



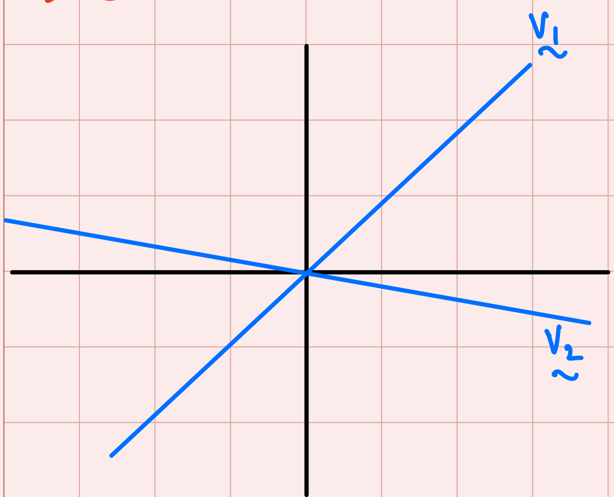


The phase plane (nonlinear $n = 2$)

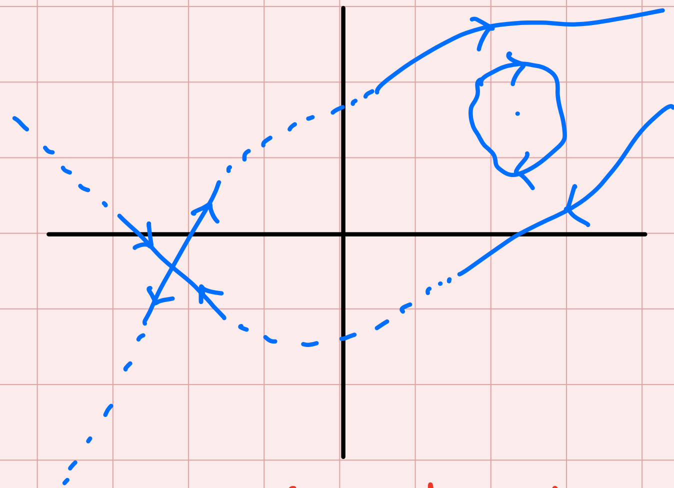
$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$$

$$x_1 = f_1(x_1, x_2)$$
$$x_2 = f_2(x_1, x_2)$$

Linear



Nonlinear



n eigenvectors paint a global picture of the phase plane.

n eigenvectors do not capture global picture.

Features of phase plane:

→ Fixed points   equilibrium solutions $f(\underset{\sim}{x}^*) = \underset{\sim}{0}$

→ Closed orbits   periodic solutions $\underset{\sim}{x}(t+T) = \underset{\sim}{x}(t)$

→ Behavior of solutions near fixed pts & closed orbits

(solutions: aka trajectories in this context.)

(near fixed pts → Linearize!)

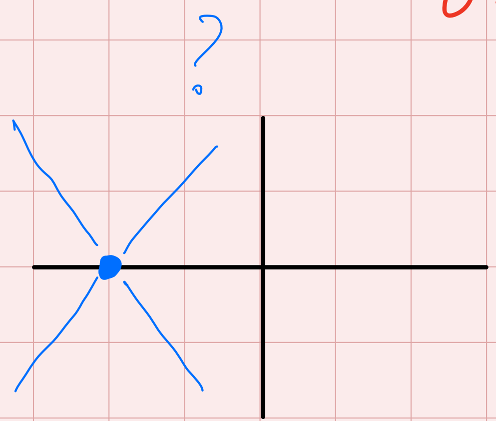
Consider $\dot{x} = x + e^{-y}$
$\dot{y} = -y$

$e^{-y} = 1 - y + \frac{y^2}{2!} - \frac{y^3}{3!} + \cdots$

$e^{-y} \approx 1 - y$

- Find fixed pt: $0 = x + e^{-y}$, $0 = -y$ $\Rightarrow$ $y = 0$, $x = -1$

$(x^*, y^*) = (-1, 0)$



$\dot{x} \approx x + 1 - y$
$\dot{y} = -y$

Define $x + 1 \longrightarrow x'$ (NOT Derivative)

$\dot{x}' = x' - y$
$\dot{y} = -y$

$$\begin{bmatrix} \dot{x}' \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\tau = 0$, $\Delta = -1$ $\Rightarrow$ saddle

$\dot{x} = f_1(x, y)$
$\dot{y} = f_2(x, y)$

and $f_1(x^*, y^*) = f_2(x^*, y^*) = 0$

Let $u = x - x^*$
$v = y - y^*$

$\dot{u} = \dot{x}$
$= f_1(x^* + u, y^* + v)$

$$= \underbrace{f_1(x^*, y^*)}_{0 \text{ by def. of fixed pt.}} + u \left.\frac{\partial f_1}{\partial x}\right|_{\substack{x=x^* \\ y=y^*}} + v \left.\frac{\partial f_1}{\partial y}\right|_{\substack{x=x^* \\ y=y^*}} + \underbrace{\text{higher order terms.}}_{\text{ignore}}$$

$$\Rightarrow \boxed{\dot{u} = u \left.\frac{\partial f_1}{\partial x}\right|_{\substack{x^* \\ y^*}} + v \left.\frac{\partial f_1}{\partial y}\right|_{\substack{x^* \\ y^*}}}$$

by a similar argument,

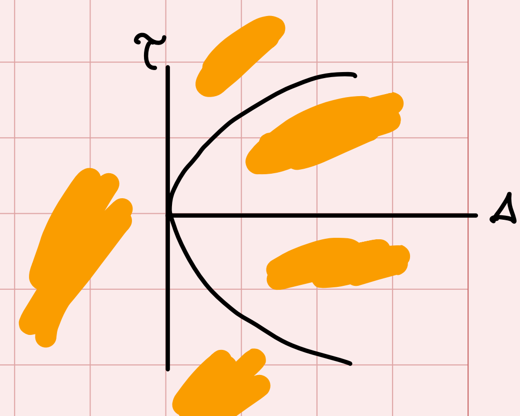
$$\boxed{\dot{v} = u \left.\frac{\partial f_2}{\partial x}\right|_{\substack{x^* \\ y^*}} + v \left.\frac{\partial f_2}{\partial y}\right|_{\substack{x^* \\ y^*}}}$$

$$A_{ij} = \frac{\partial f_i}{\partial x_j}$$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}_{\substack{x=x^* \\ y=y^*}}}_{} \cdot \begin{bmatrix} u \\ v \end{bmatrix}$$

Jacobian matrix (A) for the system $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ evaluated at $\underset{\sim}{x}^*$.

For saddles, spirals & nodes the system $\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix}$ (evaluated at (x^*, y^*)) is a good representation of the nonlinear system $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ near (x^*, y^*)



For other types of fixed points, the system $\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix}$ gives a questionable representation of the nonlinear system $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ near (x^*, y^*)

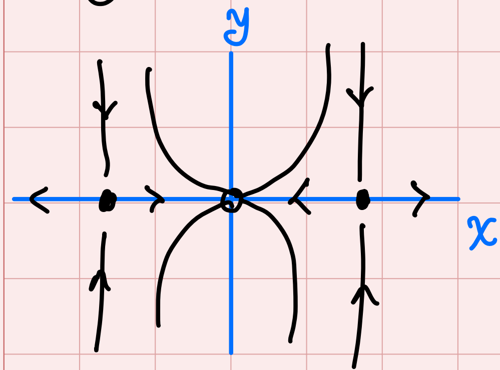
$$\dot{x} = -x + x^3$$
$$\dot{y} = -2y$$

1) Find fixed pts.
2) Characterize each.

What kind of fixed pt. is it?

calculate Jacobian matrix, evaluate at each fixed pt.

where is this matrix on τ-Δ plane?

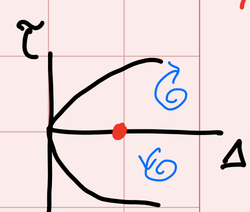


Mon, Feb 24 Lecture 10

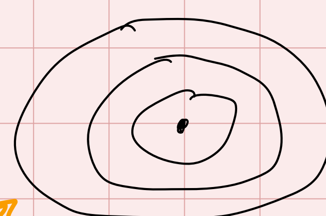
Ex.
$$\dot{x} = -y + ax(x^2 + y^2)$$
$$\dot{y} = x + ay(x^2 + y^2)$$

→ (0,0) is a fixed pt. Classify its stability

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} \text{ evaluate at } (0,0) \rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



Linear theory predicts



$$\dot{x} = -y + ax(x^2+y^2)$$
$$\dot{y} = x + ay(x^2+y^2)$$

Near the origin, looks like a center.

But — notice the axes

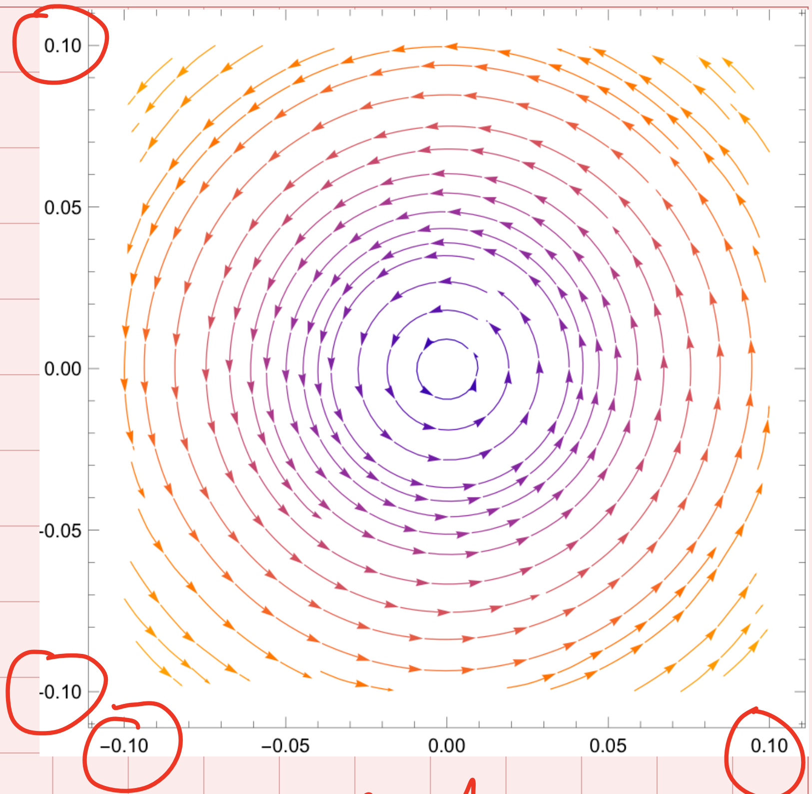
... zoom out:

and it looks like an unstable spiral.

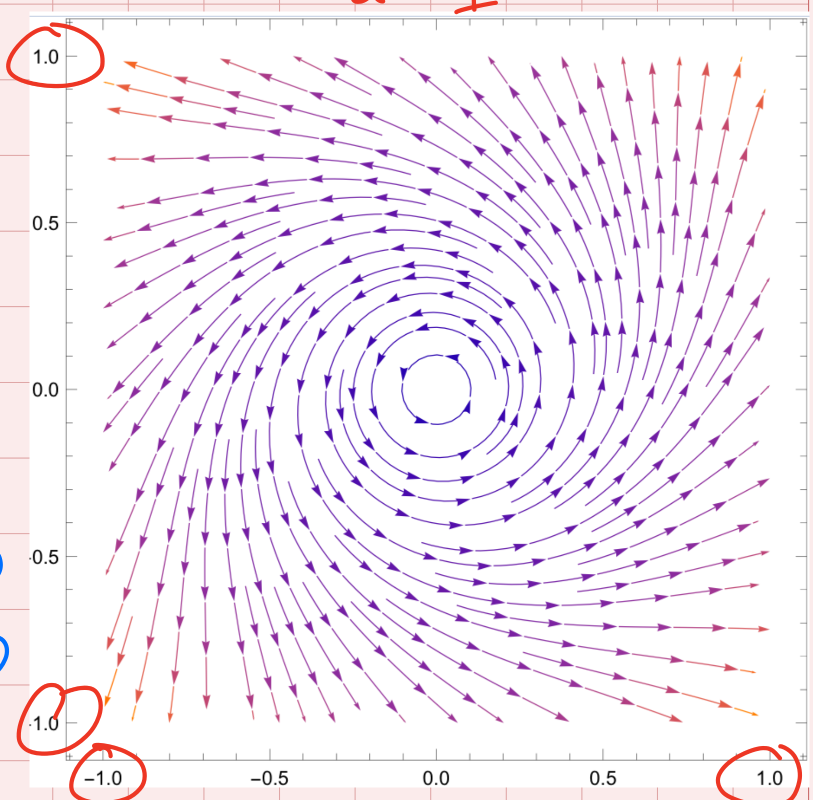
Strogatz 6.3.2 proves that this system is like an unstable spiral if $a > 0$, stable spiral if $a < 0$

(using r, θ coordinates)

$a = 1$



$a = 1$



Hyperbolic Fixed Pts.

Fixed pts that remain unchanged, qualitatively, by small nonlinear terms, relative to their linearized phase portraits.

"Local phase portrait near a hyperbolic fixed pt. is topologically equivalent to the phase portrait of its linearized version"

→ a homeomorphism exists between the two.

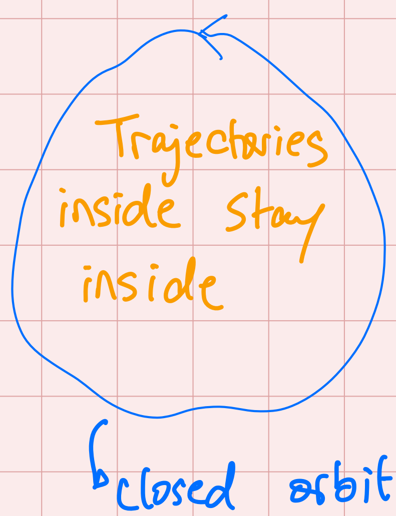
For hyperbolic fixed pts, all eigenvalues have non-zero real part.

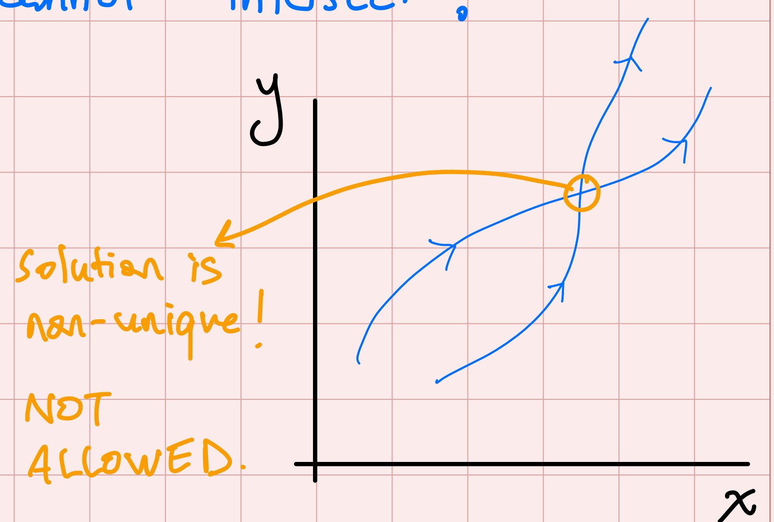
↳ if one or more eigenvalue has zero real part, the fixed pt. is nonhyperbolic.

$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}) \quad , \quad \underset{\sim}{x}(0) = \underset{\sim}{x}_0 \qquad \underset{\sim}{x} \in \mathbb{R}^n$$

if f is continuous and all partial derivatives of f are continuous on a subset $D \subset \mathbb{R}^n$, then for $\underset{\sim}{x}_0$ in D, the I.V.P above has a unique solution $\underset{\sim}{x}(t)$ at least for some time.

$\Longrightarrow$ Trajectories cannot intersect!





Lotka-Volterra Population Dynamics

- Two species competing for a resource (limited)
- Each species has a growth rate, carrying capacity

  ↳ $\dot{N} = rN(1 - N/K)$ logistic eqn.

- Two logistic eqns + competition.

x: rabbits

y: sheep

rabbits grow faster than sheep

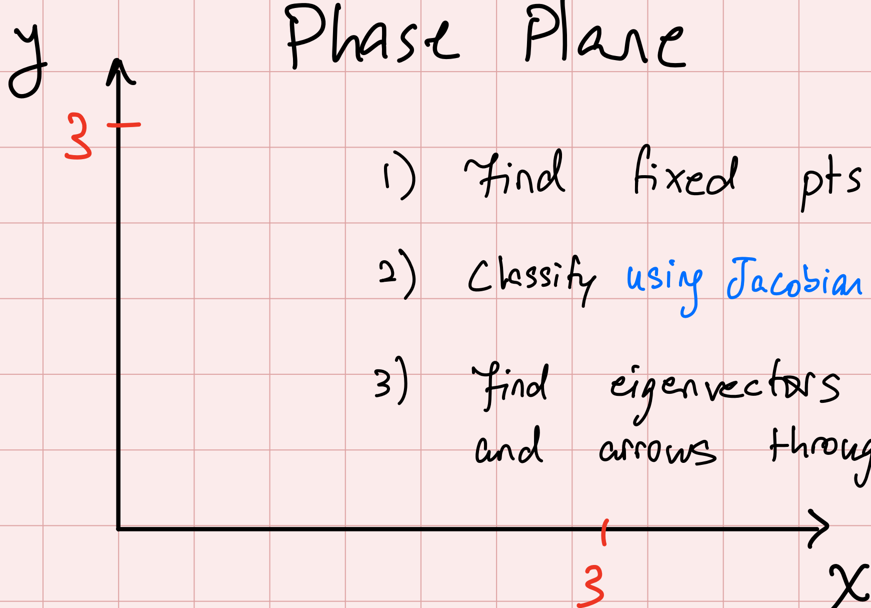
$$\dot{x} = x\left(3 - x/2 - 2y\right)$$

$$\dot{y} = y\left(2 - x - y\right)$$

Rabbits have higher carrying capacity

Sheep stronger than rabbits.

Phase Plane



1) Find fixed pts (0,2) (0,0) (6,0) (2/3, 4/3)
2) Classify using Jacobian
3) Find eigenvectors to put lines and arrows through the fixed pts.

Wed, Feb 26 Lecture 11

Building up nonlinear phase plane using linearizations at each fixed pt.

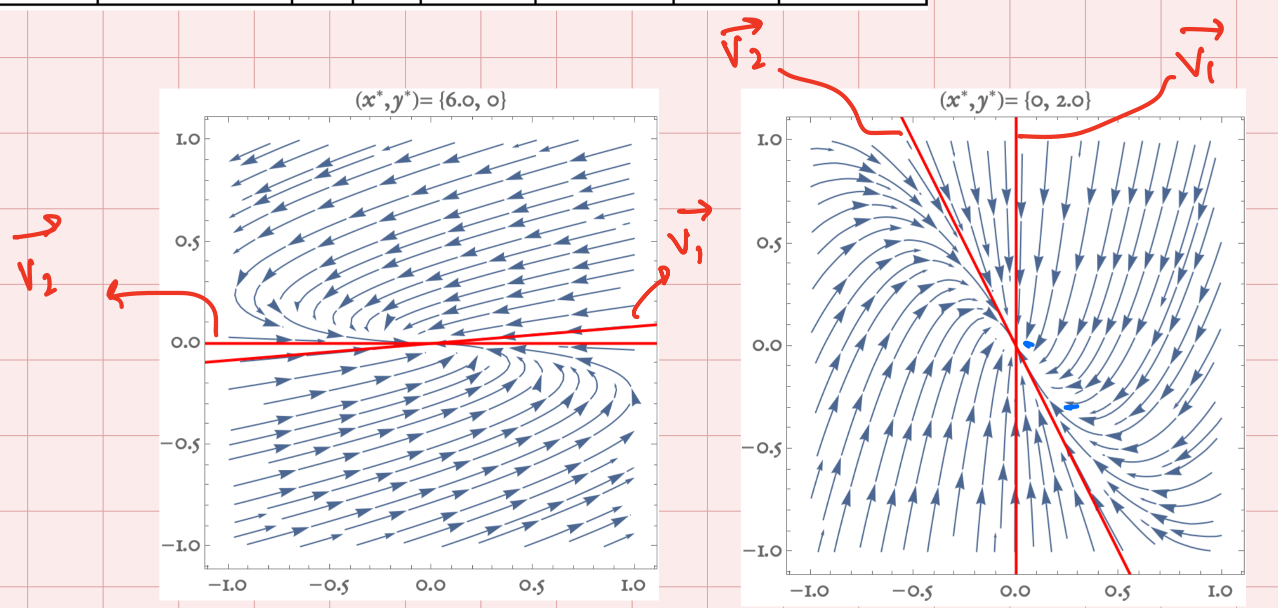
$(0,2) \longrightarrow \quad A = [\quad] \quad \lambda_{1,2} = \dots \quad \vec{v}_{1,2} = [\ ]$

$(6,0) \longrightarrow$

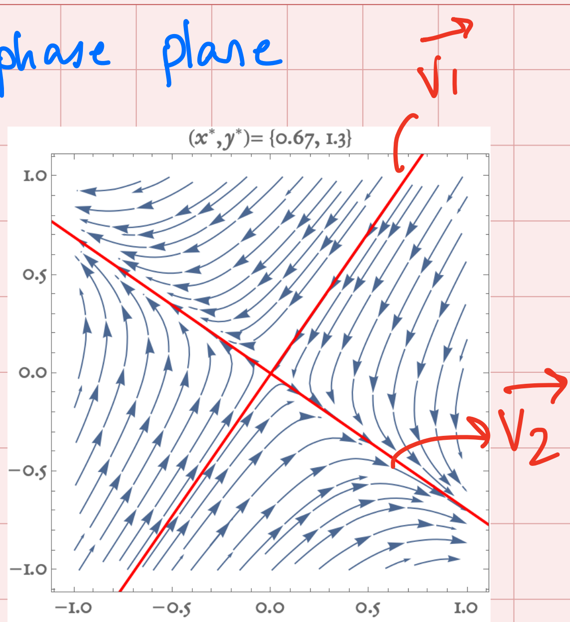
$(0,0) \longrightarrow$

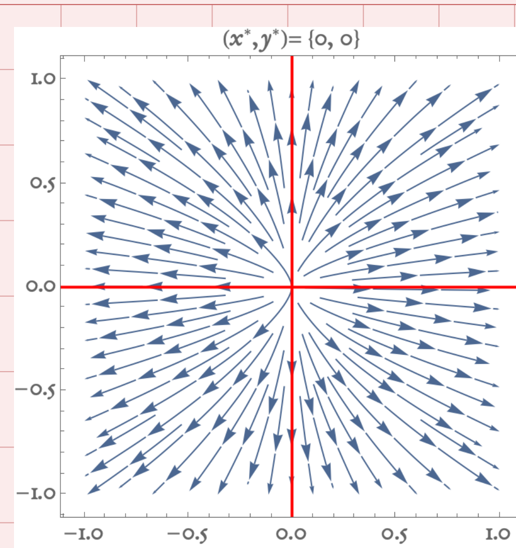
$(2/3, 4/3) \longrightarrow \quad \lambda = \{-2.26, +0.591\}$

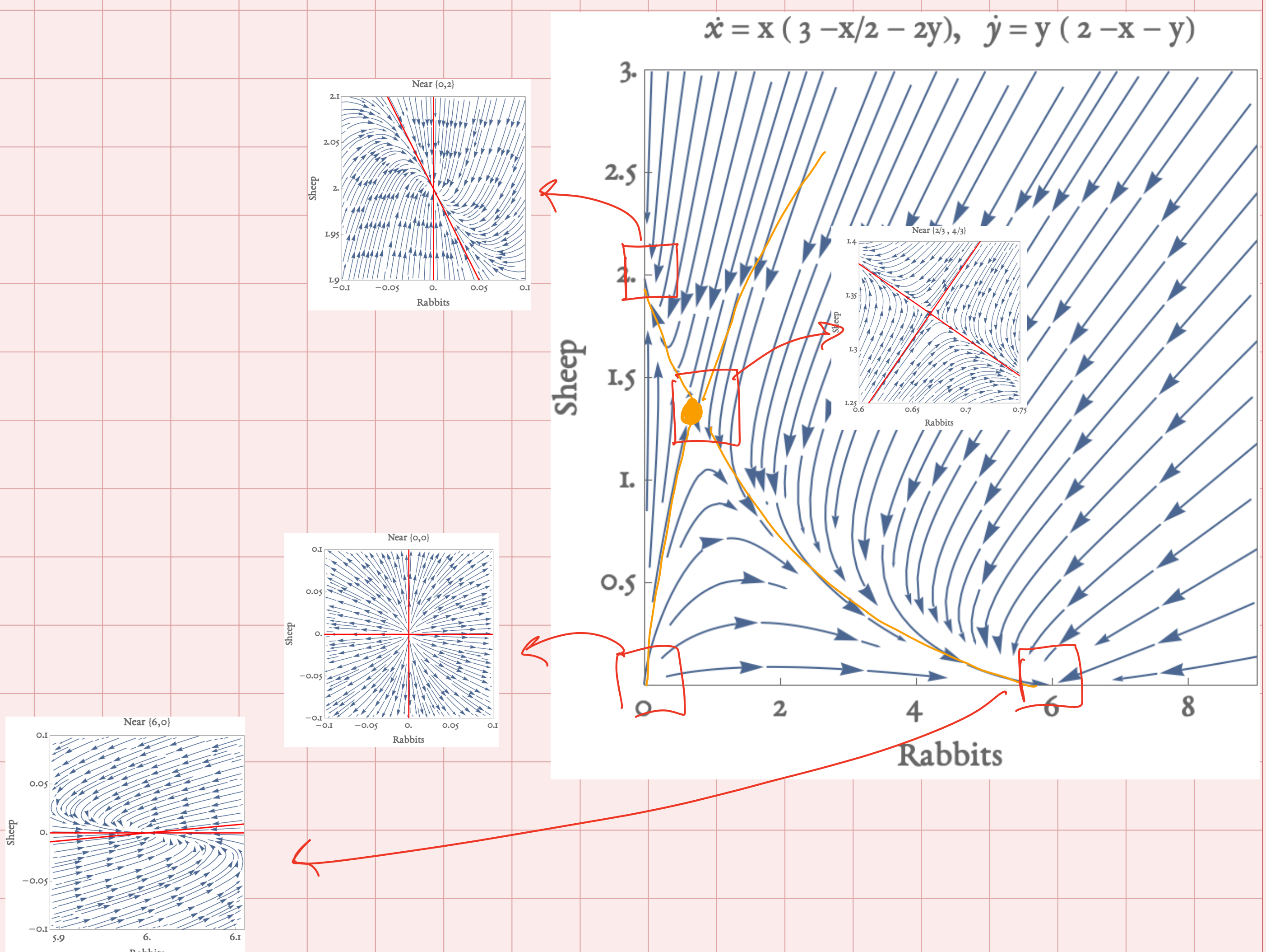
Fixed Point	Jacobian	τ	Δ	λ_1	$\vec{v}_1$	λ_2	$\vec{v}_2$	
$\begin{pmatrix}0\\2\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\-2 & -2\end{pmatrix}$	-3	2	-2.00	$\begin{pmatrix}0\\1.00\end{pmatrix}$	-1.00	$\begin{pmatrix}-1.00\\2.00\end{pmatrix}$	stable node
$\begin{pmatrix}\frac{2}{3}\\\frac{4}{3}\end{pmatrix}$	$\begin{pmatrix}-\frac{1}{3} & -\frac{4}{3}\\-\frac{4}{3} & -\frac{4}{3}\end{pmatrix}$	$-\frac{5}{3}$	$-\frac{4}{3}$	-2.26	$\begin{pmatrix}0.693\\1.00\end{pmatrix}$	0.591	$\begin{pmatrix}-1.44\\1.00\end{pmatrix}$	saddle
$\begin{pmatrix}6\\0\end{pmatrix}$	$\begin{pmatrix}-3 & -12\\0 & -4\end{pmatrix}$	-7	12	-4.00	$\begin{pmatrix}12.0\\1.00\end{pmatrix}$	-3.00	$\begin{pmatrix}1.00\\0\end{pmatrix}$	stable node
$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}3 & 0\\0 & 2\end{pmatrix}$	5	6	3.00	$\begin{pmatrix}1.00\\0\end{pmatrix}$	2.00	$\begin{pmatrix}0\\1.00\end{pmatrix}$	unstable node



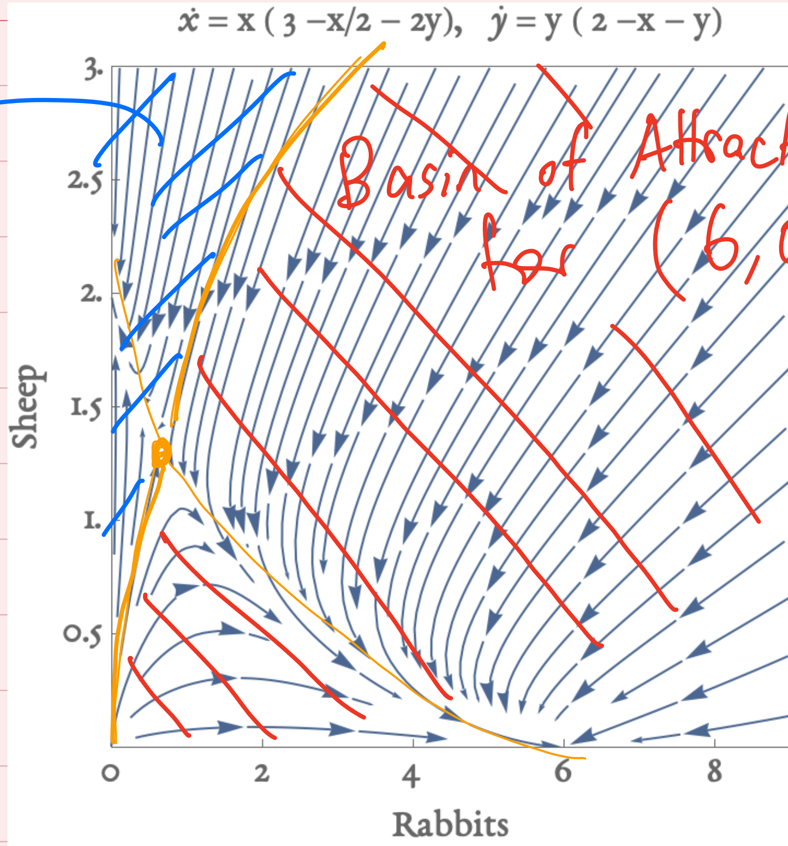
Linear phase plane







Basin of Attraction for (0,2)



Conservative Systems

Given $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$, $\underset{\sim}{x} \in \mathbb{R}^n$, a conserved quantity is a real-valued continuous (scalar) function $E(\underline{x}) \neq$ const. that is constant on trajectories, i.e. $\frac{dE}{dt} = 0$ (chain rule)

If a system has a conserved quantity, it is called a conservative system.

$$\underbrace{\ddot{x}}_{\text{"accel."}} = \underbrace{x^3 - x}_{\text{"force" } F(x)}$$

Find a conserved quantity for this system.

$$F(x) = -\frac{dV}{dx}$$

potential $V(x)$

$$x^3 - x = -\frac{dV}{dx}$$

$$\ddot{x} = F(x) = -dV/dx$$

$$\int (x^3 - x)\,dx = \int -\,dV$$

$$\ddot{x} + \frac{dV}{dx} = 0$$

$$\frac{x^4}{4} - \frac{x^1}{2} = -V + c$$

$$\dot{x}\ddot{x} + \dot{x}\frac{dV}{dx} = 0$$

$$\Rightarrow \boxed{V(x) = \frac{x^2}{2} - \frac{x^4}{4} + c}$$

$$\frac{d}{dt}\Big(\underbrace{\frac{1}{2}\dot{x}^2 + V(x)}_{E(x,\dot{x})}\Big) = 0$$

we have found that $\frac{1}{2}\dot{x}^2 + V(x)$ is a conserved quantity.

① $m\ddot{x} + c\dot{x} + kx = 0$

② $\ddot{\theta} = \sin\theta$

↓

None exists

Find $V(x)$, not $V(x,\dot{x})$

$$m\ddot{x} + kx = 0$$

$$E = \frac{1}{2}\dot{\theta}^2 + \cos\theta$$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

Mon, Mar 3 Lecture 12

When are systems conservative?

↳ Physically, they correspond to frictionless mechanical systems + others

↳ Mathematically, when you can find $E(\underline{x})$.

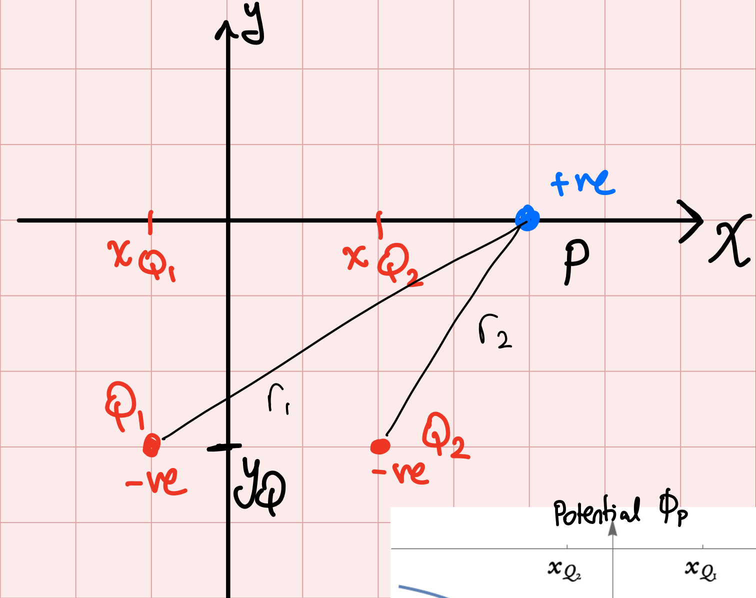
Ex

A positive charge P is confined to move on x-axis.

$$r_1 = \sqrt{(x - x_{Q_1})^2 + y_Q^2}$$

Location of P

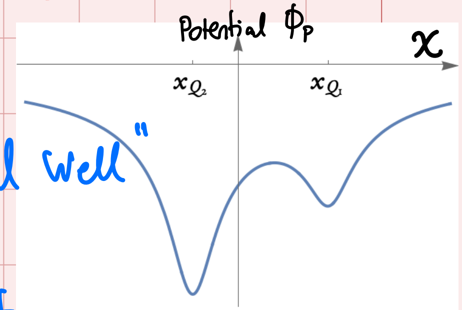
$$r_2 = \sqrt{(x - x_{Q_2})^2 + y_Q^2}$$



Electric Potential

$$\phi_p = \frac{Q_1}{r_1} + \frac{Q_2}{r_2}$$

"Potential Well"



$$\text{Force} = -\nabla \phi$$

$$= \frac{Q_1}{r_1^2}\hat{e}_{r_1} + \frac{Q_2}{r_2^2}\hat{e}_{r_2} \rightarrow \text{horz component}$$

Horz force: $\dfrac{Q_1}{(x - x_{Q_1})^2 + y_Q^2} \dfrac{x - x_{Q_1}}{\sqrt{(x-x_{Q_1})^2 + y_Q^2}} + \dfrac{Q_2}{(x - x_{Q_2})^2 + y_Q^2} \dfrac{x - x_{Q_2}}{\sqrt{(\dots)}}$

" $\ddot{x}$ = force "

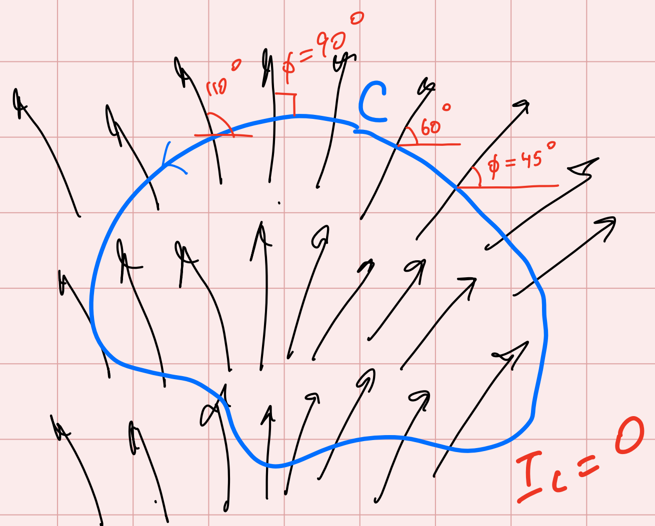
Index of a closed curve in a vector field.

A Measure of the "winding" of the vector field.

Let $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$, $\underset{\sim}{x} \in \mathbb{R}^2$ and C a closed curve

(a smooth vector field)

(does not self-intersect)
(does not pass thru fixed pts of the vector field)



Let ϕ be the angle between the vector field $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}$ and the horizontal for any point on C.

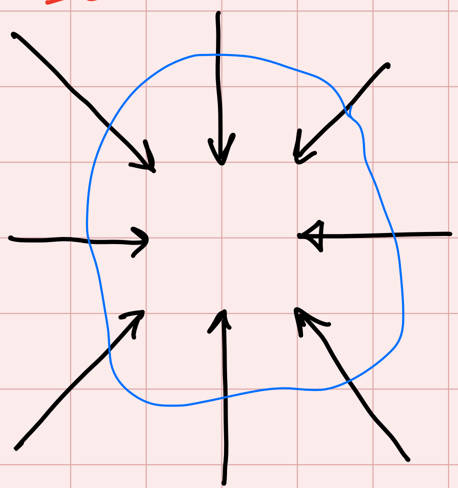
$\Rightarrow \phi = \tan^{-1}(\dot{y}/\dot{x})$

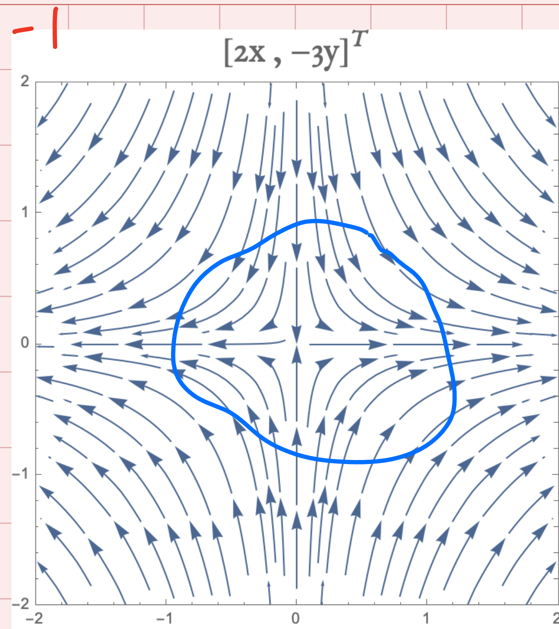
$[\phi]_c$: net change in ϕ over one c.c.w. loop around C.

Index of C is:

$$I_c = \frac{1}{2\pi}[\phi]_c$$

→ net number of counter-clockwise revolutions made by the vector field $f(\underset{\sim}{x})$ as $\underset{\sim}{x}$ moves counterclockwise around C.





Spirals, centers, stars, degenerate nodes : $I_c = +1$

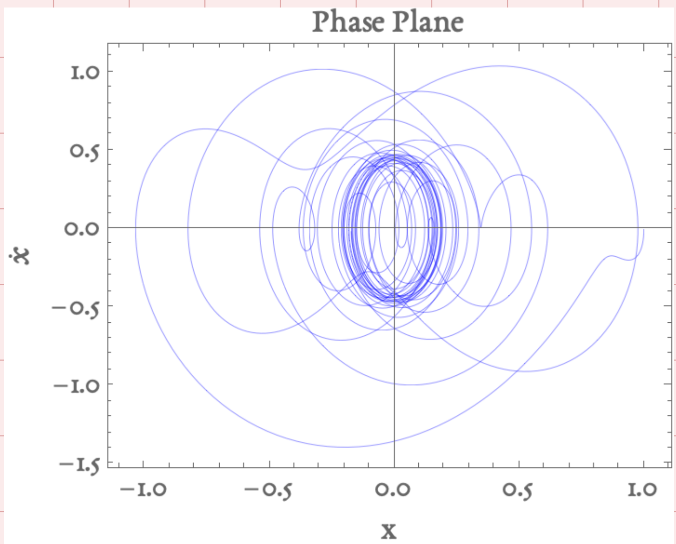
Saddles $I_c = -1$

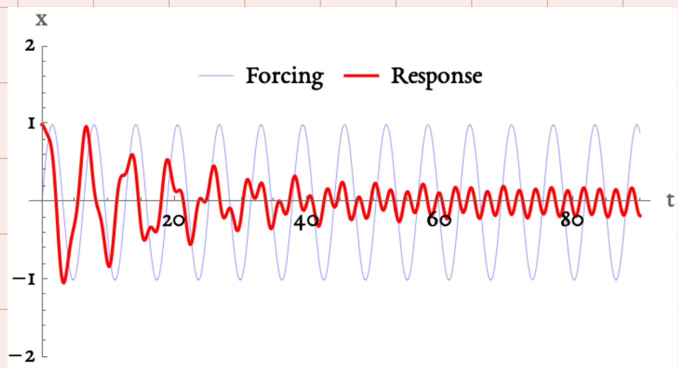
- $I_c = 0$ if no fixed pts inside C.
- If C is a trajectory, $I_c = +1$
- If C can be continuously deformed into C' without passing thru a fixed pt, $I_c = I_{c'}$.
- If all arrows in the vector field reverse direction $\vec{f} \to -\vec{f}$ then I does not change.
- The index of a fixed point $= I_c$ for any C that encloses only that fixed pt.
- Any closed orbit in phase plane must enclose fixed pts whose indices sum up to $+1$.
- If C surrounds multiple fixed pts, $I_c = \sum$ index of each fixed pt.

Wed, Mar 5 Lecture 13

Limit Cycles

$$m\ddot{x} + c\dot{x} + kx = \sin(\omega t)$$





This is not an $n=2$ dynamical system $\{\dot{\underline{x}} = f(\underline{x})\}$

An n^{th} order time-dependent (= nonautonomous) eqn. is a special case of an $(n+1)^{th}$ order autonomous dynamical system

$$\ddot{x} + \dot{x} + x = \sin \omega t$$

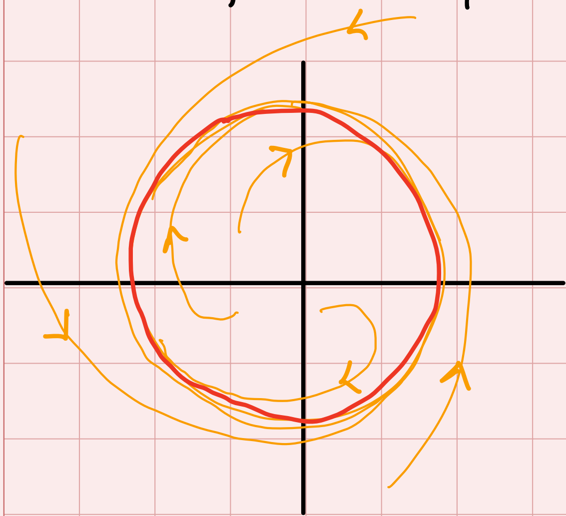
$$x_1 = x \qquad x_2 = \dot{x} \qquad x_3 = \omega t$$

$$\dot{\underline{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - x_2 + \sin x_3 \\ \omega \end{bmatrix}$$

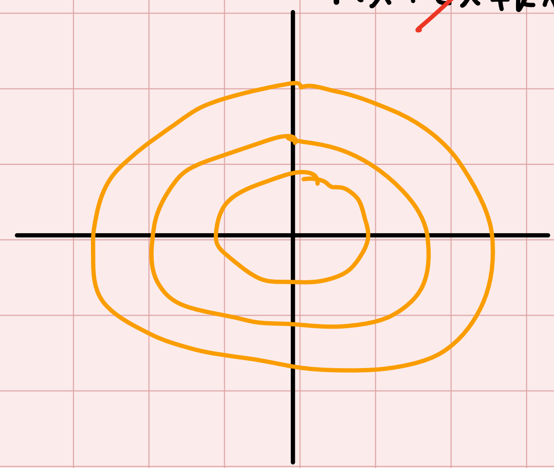
Limit Cycle is an isolated closed trajectory in phase space

(closed orbit)

$m\ddot{x} + c\dot{x} + kx = 0$ ($c \to 0$)

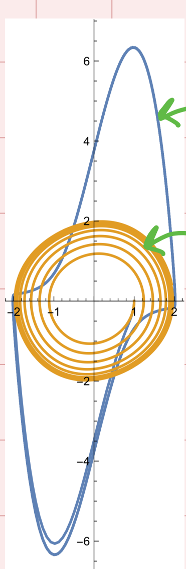


closed orbit that is a limit cycle.



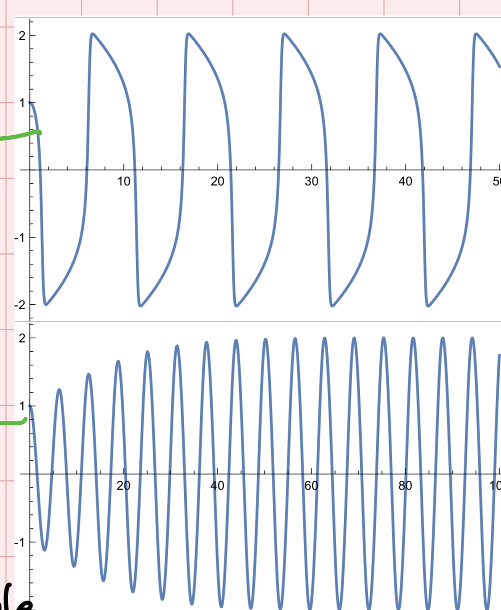
closed orbits but NOT limit cycles

<u>van der Pol oscillator</u> $\ddot{x} + \mu(x^2-1)\dot{x} + x = 0$



2 diff limit cycles

A distinctive feature of nonlinear systems



All trajectories approach the limit cycle (b/c it's <u>stable</u>). Opposite for unstable ones.

Proving the absence of closed orbits

(closed orbits: a superset of limit cycles.)

1) Gradient Systems approach.

If a system $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x})$ can be written as $\dot{\underset{\sim}{x}} = -\nabla V(\underset{\sim}{x})$ for some scalar function $V(\underset{\sim}{x})$, then no closed orbits exist for this system. → continuously differentiable, single-valued.

Recall: in 1-d, $\dot{x} = f(x)$, we could always find $V(x)$ such that $f(x) = -dV/dx$ → no oscillations in 1-d.

in 2-d, not so sure; hard to find V

$$\ddot{\theta} + \sin\theta = 0$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\sin x_1$$

$$\underset{\sim}{f}(\underset{\sim}{x}) = \begin{bmatrix} x_2 \\ -\sin x_1 \end{bmatrix}$$ no V can be found.

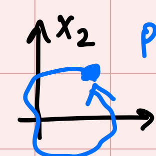
look for V, s.t. $\underset{\sim}{f} = -\nabla V$

$$-\frac{\partial V}{\partial x_1} = x_2$$

$$-\frac{\partial V}{\partial x_2} = -\sin x_1$$

cannot establish the absence of closed orbits.

Proof $\dot{\underset{\sim}{x}} = -\nabla V(\underset{\sim}{x})$



change in V from $t=0$ to $t=T$. → must be 0.

$$\Delta V = \int_0^T dV = \int_0^T \frac{dV}{dt}\cdot dt$$

$$\frac{dV}{dt} \to \frac{dV}{d\underset{\sim}{x}}\cdot\frac{d\underset{\sim}{x}}{dt} = (\nabla V)\cdot\dot{\underset{\sim}{x}}$$

$$\int_0^T (\nabla V \cdot \dot{\underset{\sim}{x}})\, dt$$

$$= \int_0^T (-\dot{\underset{\sim}{x}}\cdot\dot{\underset{\sim}{x}})\, dt = -\int_0^T \|\dot{\underset{\sim}{x}}\|^2 dt < 0 \quad \text{CONTRAD.}$$

Given a system $\dot{\underset{\sim}{x}} = -\nabla V(\underset{\sim}{x})$ a closed orbit is impossible

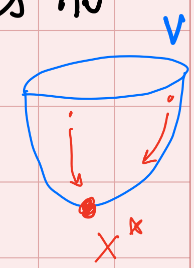
2) Liapunov Function approach

$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ with $f(\underset{\sim}{x}^*) = \underset{\sim}{0}$

If we can find a continuously differentiable real-valued function $V(\underset{\sim}{x})$ that:

$$\left.\begin{array}{l} V(\underset{\sim}{x}) > 0 \quad \forall\ \underset{\sim}{x} \neq \underset{\sim}{x}^* \\ V(\underset{\sim}{x}^*) = 0 \\ \dot{V}(\underset{\sim}{x}) < 0 \quad \forall\ \underset{\sim}{x} \neq \underset{\sim}{x}^* \end{array}\right\}$$

then system has no closed orbits.



where $V(\underset{\sim}{x})$: "an energy-like function that decreases along trajectories."

3) Dulac's Criterion

$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ and f is defined on $D \subset \mathbb{R}^2$: a simply connected subset.

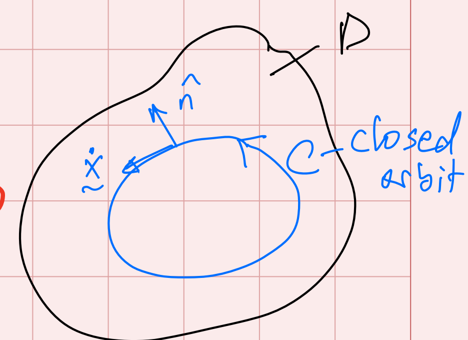
If there exists $g(\underset{\sim}{x})$ such that $\nabla \cdot (g\dot{\underset{\sim}{x}})$ has one sign throughout D, then there are no closed orbits entirely within D.

$$\iint_A \nabla \cdot (g\dot{\underset{\sim}{x}})\, dA = \oint g\dot{\underset{\sim}{x}} \cdot \hat{n}\, ds$$

"has one sign" $\Rightarrow$ term $\neq 0$

$\oint g\dot{\underset{\sim}{x}} \cdot \hat{n}\, ds = 0$ since $\dot{\underset{\sim}{x}} \cdot \hat{n} = 0$

$\Rightarrow$ No C exists



Presence of Closed orbits

Poincaré–Bendixson Thm

If

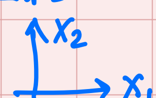
1. D is a closed, bounded subset of $\mathbb{R}^2$.
2. $\dot{\underline{x}} = f(\underline{x})$ is defined on some open set that includes D.
3. D does not have any fixed pts.
4. There exists a trajectory that is confined in D: starts in D and stays inside D for all future time

"trapping region"

Then C is a closed orbit or approaches a closed orbit.

implications for Chaos

The topology of $\mathbb{R}^2$ prevents anything too wild from happening in the phase plane



we know that trajectories of $\dot{\underline{x}} = f(\underline{x})$ cannot self-intersect.

If a trajectory is known to be trapped in a certain finite subset of $\mathbb{R}^2$, it must eventually settle down into a limit cycle.

It cannot keep wandering forever: it will eventually run out of room.

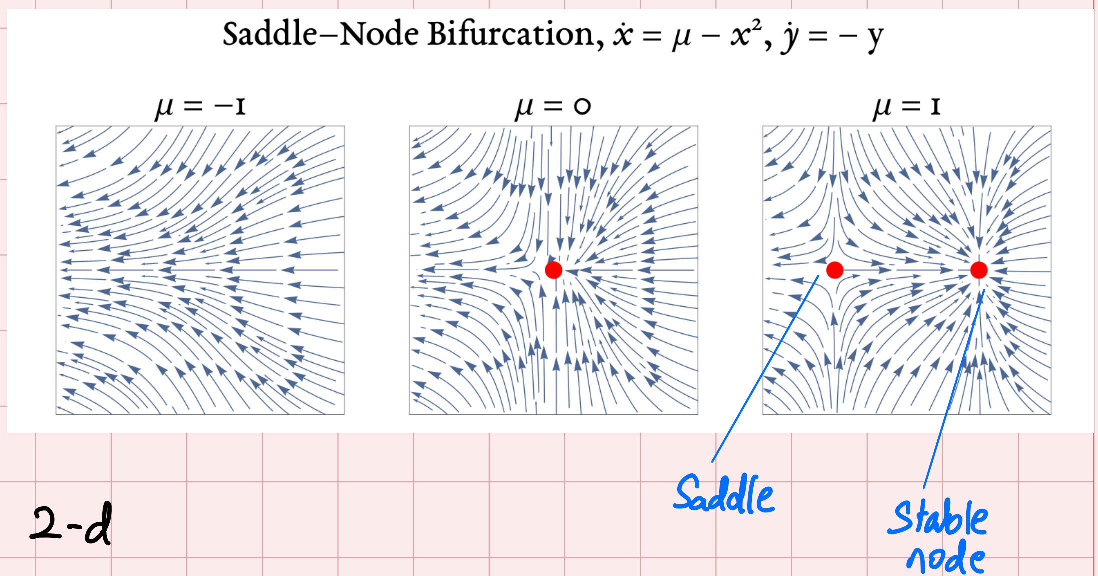
In $n \geq 3$ autonomous systems, chaos is possible because $\mathbb{R}^3$ has infinitely more room than $\mathbb{R}^2$. If $n \geq 3$, a trajectory can be confined to a subset of $\mathbb{R}^n$ and wander around forever without self-intersecting. ⟶ STRANGE ATTRACTORS.

Mon, Mar 24 Lecture 15

Bifurcations

A bifurcation occurs when the topology of phase space changes (qualitatively)

$$\dot{x} = \mu - x^2$$
$$\dot{y} = -y$$

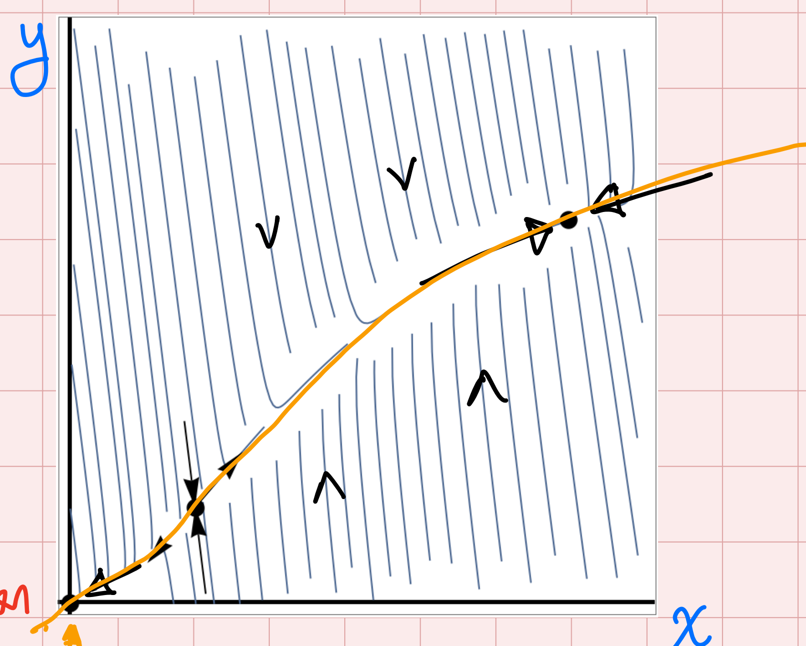


Saddle-node Bifurcation in 2-d occurs in one dimension

$$\dot{x} = -ax + y$$
$$\dot{y} = \frac{x^2}{1+x^2} - by$$

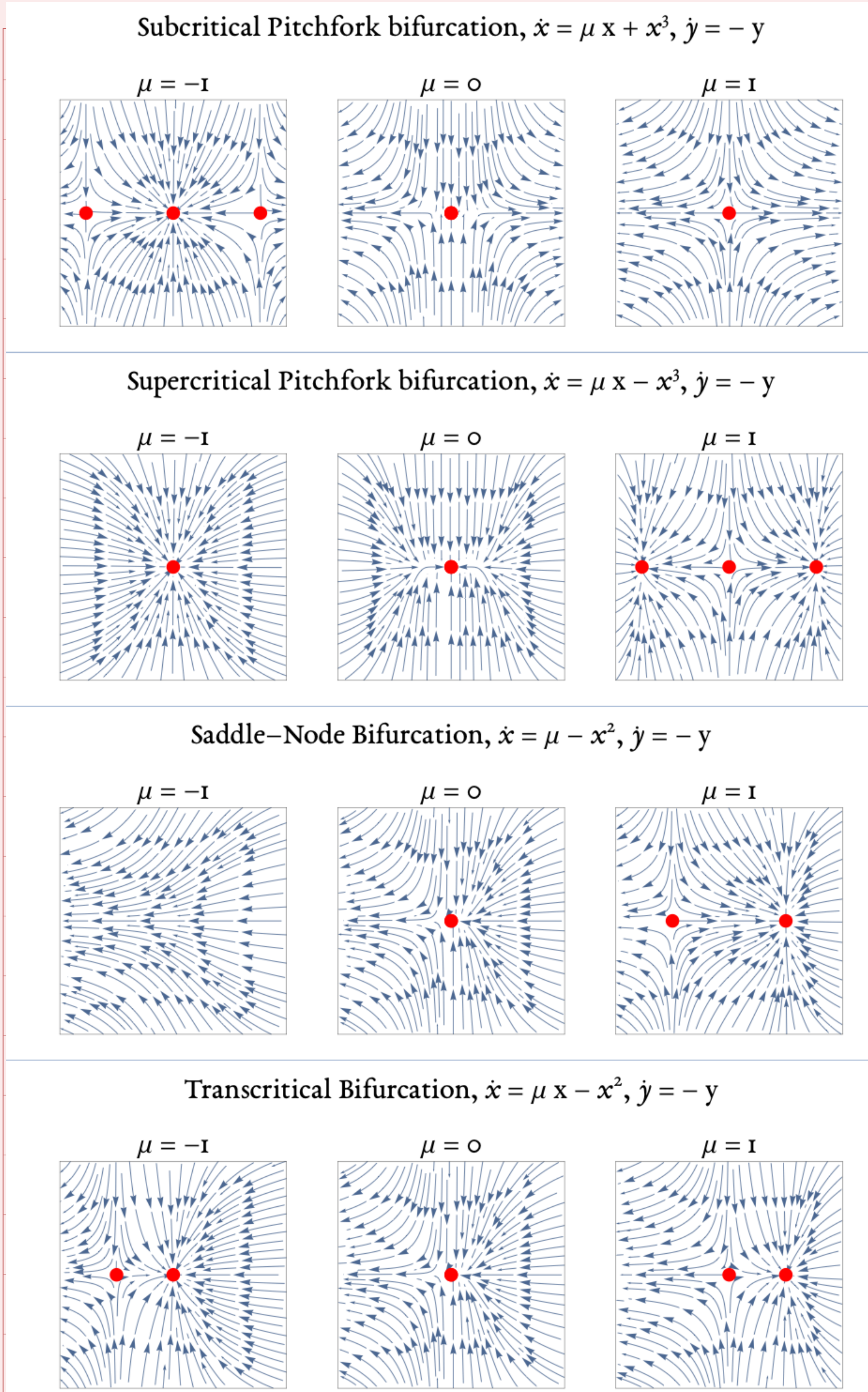
$a, b > 0$

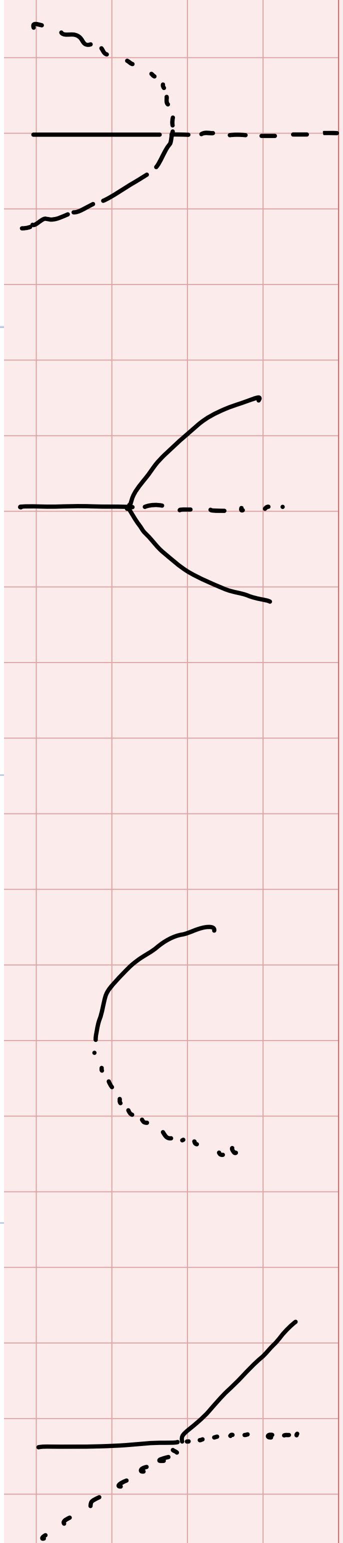
$x, y > 0$

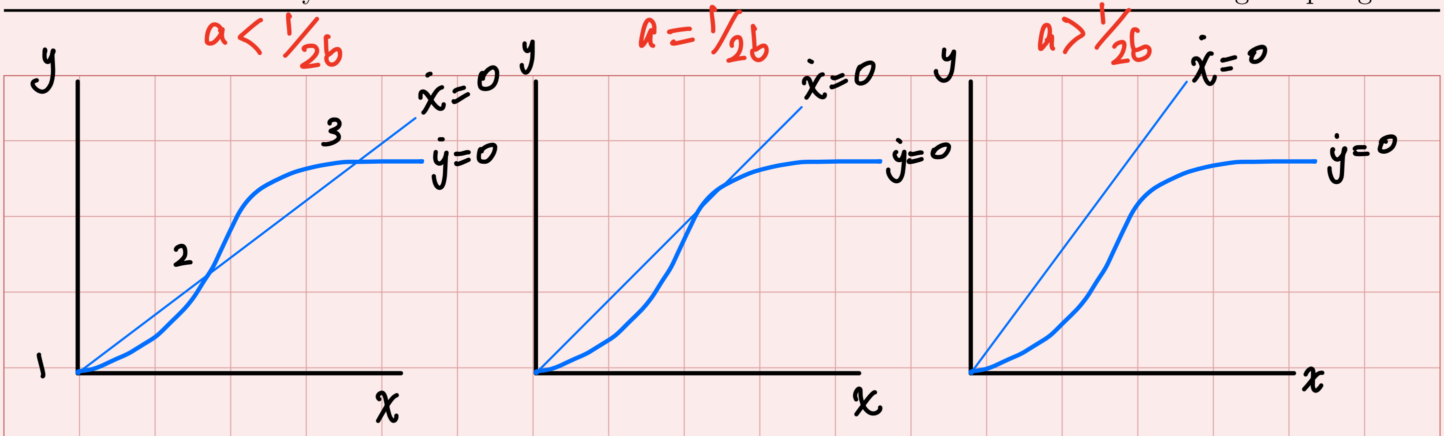


in this system, bifurcation occurs on this curve

tinyurl.com/E91bifurcations2d

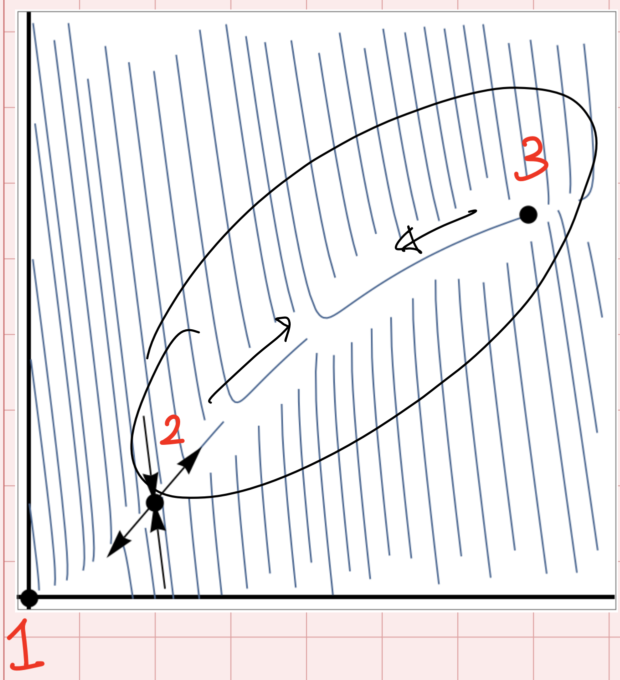

Subcritical Pitchfork bifurcation, $\dot{x} = \mu x + x^3$, $\dot{y} = -y$
$\mu = -1$
$\mu = 0$
$\mu = 1$
Supercritical Pitchfork bifurcation, $\dot{x} = \mu x - x^3$, $\dot{y} = -y$
$\mu = -1$
$\mu = 0$
$\mu = 1$
Saddle–Node Bifurcation, $\dot{x} = \mu - x^2$, $\dot{y} = -y$
$\mu = -1$
$\mu = 0$
$\mu = 1$
Transcritical Bifurcation, $\dot{x} = \mu x - x^2$, $\dot{y} = -y$
$\mu = -1$
$\mu = 0$
$\mu = 1$





Fixed pts are at $ax\,b(1+x^2) = x^2$: $(0,0)$

and at $x = \dfrac{1 \pm \sqrt{1-(2ab)^2}}{2ab}$



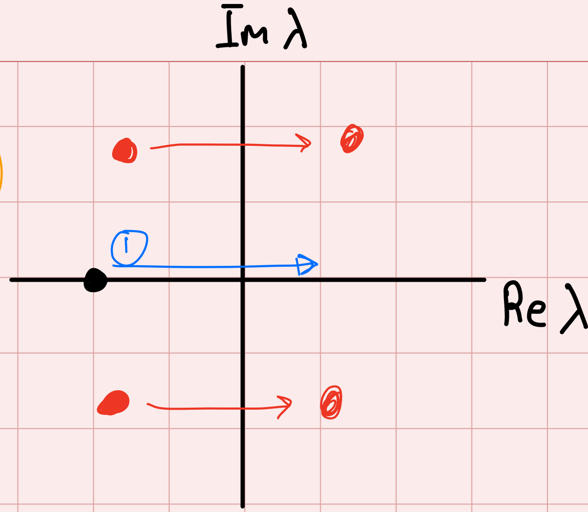
Two of the fixed points meet and annihilate each other. (2 and 3)

Example of a saddle-node bifurcation

<in-class exercise>

Wed, Mar 26 Lecture 16

Hopf Bifurcations (Limit cycles)



① A stable node becomes unstable

Bifurcations seen so far involve $\lambda = 0$ at the critical point.

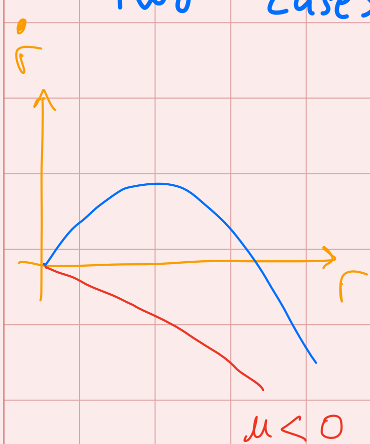
② Hopf Bifurcation

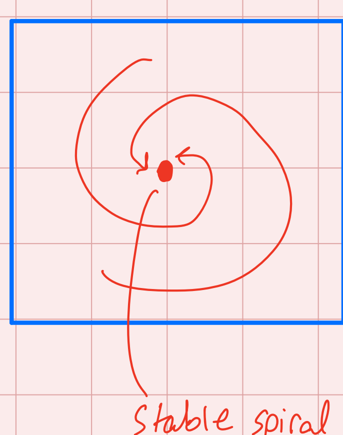
Bifurcations occur when $\mathrm{Im}(\lambda) \neq 0$

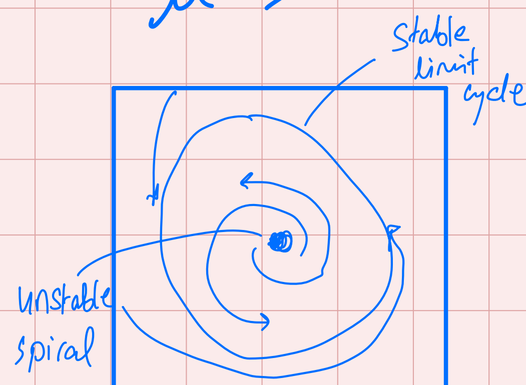
$$\dot{r} = \mu r - r^3$$
$$\dot{\theta} = \omega + b r^2$$

→ fixed points in r occur at $r = 0$, $r = \pm\sqrt{\mu}$ (only if μ is positive)

Two cases: $\mu < 0$ $\mu > 0$







convert to x, y

$$\dot{r} = \mu r - r^3$$
$$\dot{\theta} = \omega + br^2$$

$x = r\cos\theta$, $y = r\sin\theta$

$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$ plug in

$$\dot{x} = (\mu r - r^3)\cos\theta - r(\omega + br^2)\sin\theta$$
$$= \mu x - (x^2+y^2)x - y(\omega + b(x^2+y^2))$$
$$= \mu x - \omega y + \text{higher-order terms}$$

$\dot{y} = \omega x + \mu y + \dots \Rightarrow A = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}$

Eigenvalues are $\boxed{\lambda = \mu \pm i\omega}$

As $\mu \uparrow$, eigenvalues cross the imaginary axis.

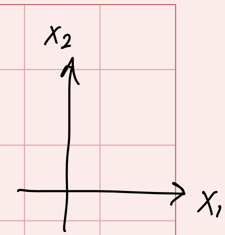
Types of Hopf Bifurcation:

- supercritical
- subcritical
- saddle-node
- ∞-period
- homoclinic

<in-class exercise>

Quasiperiodicity

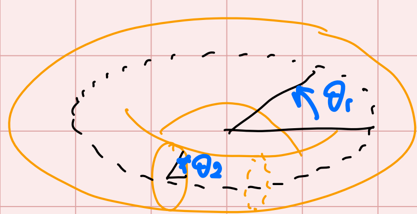
Mon, Mar 31 Lecture 17

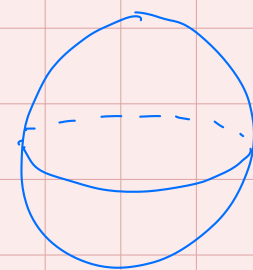


We usually think of $n=2$ dynamics on $\mathbb{R}^2$

i.e $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}$

But if the two variables can be interpreted as angles then the phase space is not the plane but is instead the surface of a torus

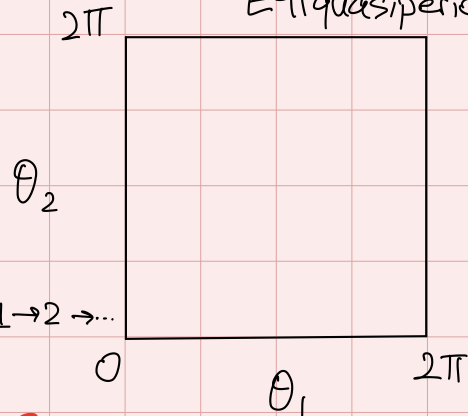




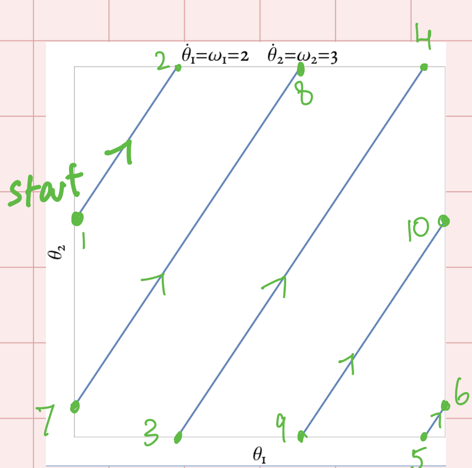
Simple $n=2$ system with periodic coordinates

$$\left.\begin{aligned} \dot\theta_1 &= \omega_1 \\ \dot\theta_2 &= \omega_2 \end{aligned}\right\} \text{ for some constant } \omega\text{'s.}$$

E91quasiperiodicity



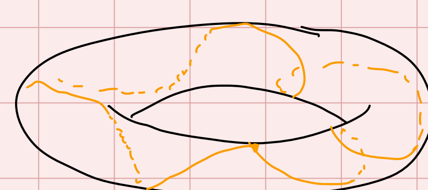
Plot trajectories for different ω's.



1 → 2 → .... → 10 → 1 → 2 → ...

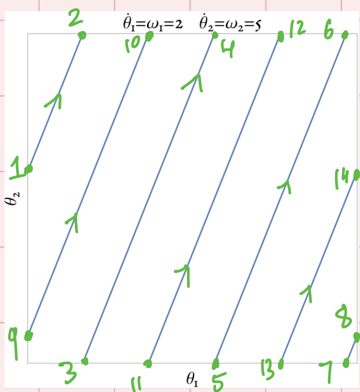
Revolutions in θ_1 :

Revolutions in θ_2 :



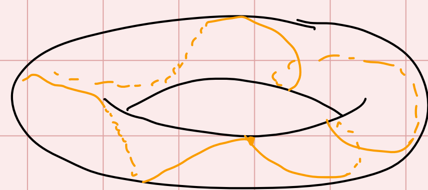
$\omega_1 = 2$ Revolutions in θ_1 : 1

$\omega_2 = 6$ Revolutions in θ_2 : 3



Rev's in θ_1 : 2

Rev's in θ_2 : 5



Trajectory goes twice around the large circle, 5x around small circle, comes back to starting point.

These were examples of periodic flow on torus.

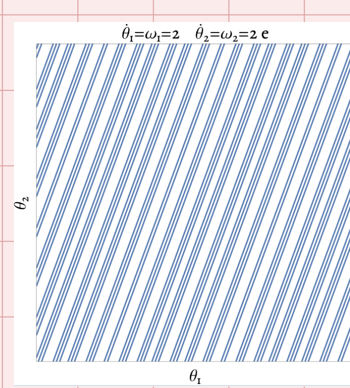
Trajectories are straight lines with slope ω_2/ω_1.

if $\omega_1/\omega_2 = p/q$ for some integers p, q

then θ_1 completes p revolutions in the time θ_2 completes q revolutions.

if ω_1/ω_2 irrational, flow in phase space is quasiperiodic; any trajectory fills the phase space without ever repeating.

tinyurl.com/E91quasiperiodicity



The Lorenz equations

$\sigma, r, b > 0$

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

Simplified model of atmospheric convection.

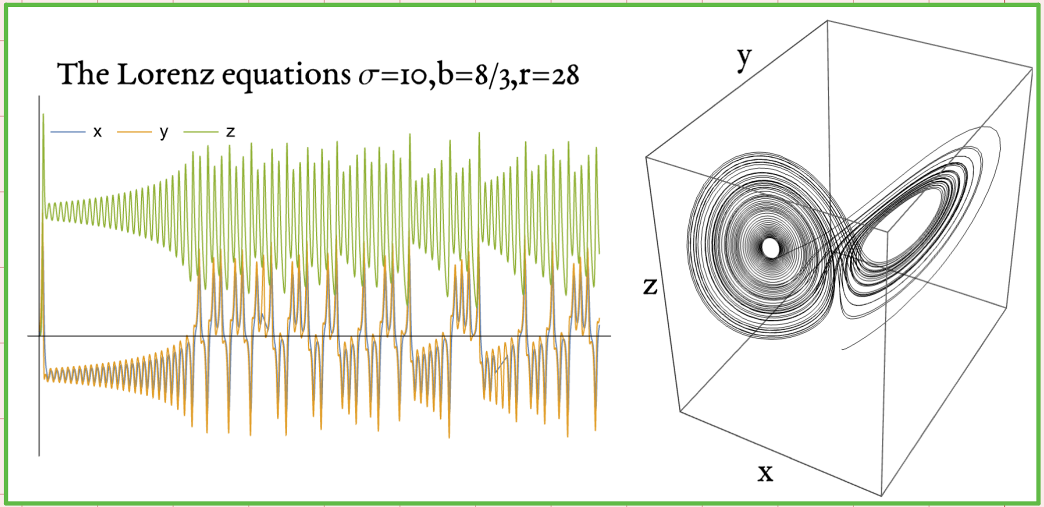
→ Nonlinear

→ 3-dimensional phase space

→ Symmetric : $(x,y,z) \longmapsto (-x,-y,z)$

tinyurl.com/E91lorenz1
tinyurl.com/E91lorenz2

Wed, Apr 2 Lecture 18

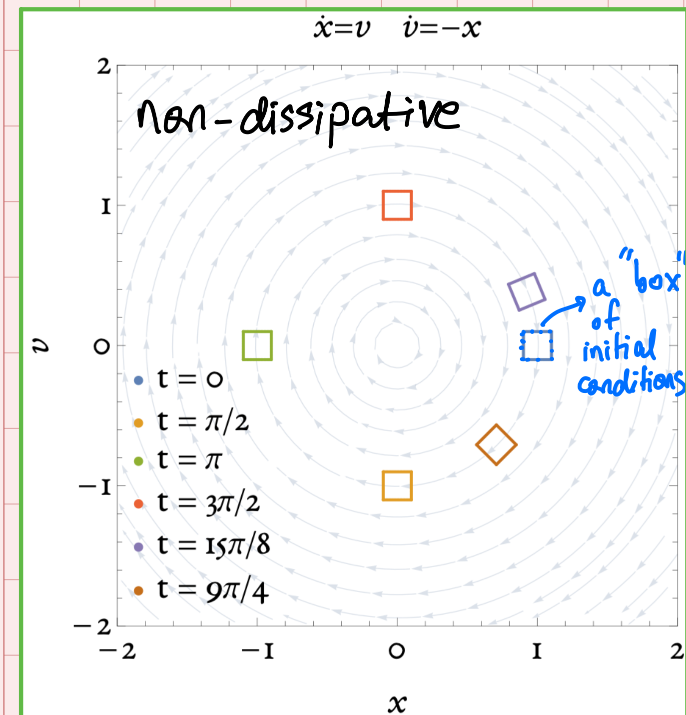


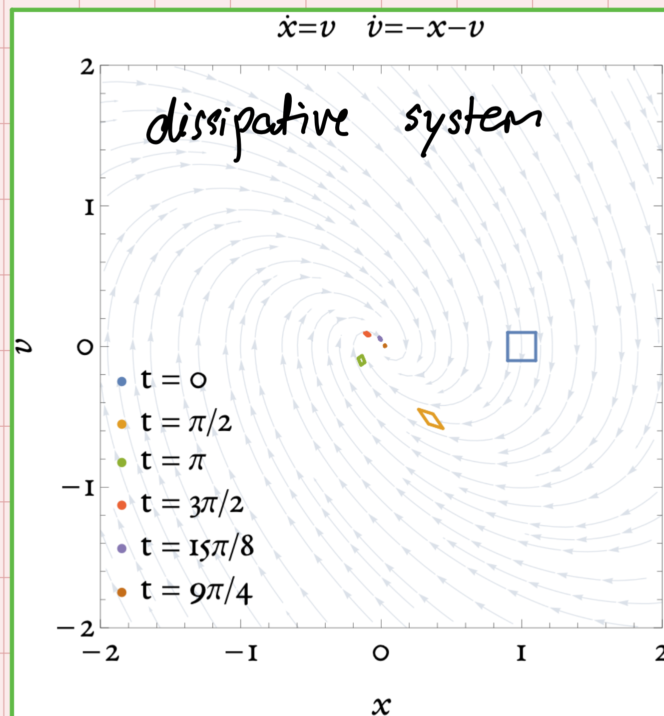
→ Dissipative : "Volumes in phase space" shrink exponentially with time

i.e. $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x})$, $\nabla \cdot \underset{\sim}{f} < 0$ for dissipative systems.

$\nabla = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix}$

If $\nabla \cdot \underset{\sim}{f} > 0$ in some region, volumes expand
If $\nabla \cdot \underset{\sim}{f} < 0$ " " " , volumes shrink





The rate of change of volumes is $\dot{V} = \int_V (\nabla \cdot \underset{\sim}{f})\, dV$

For Lorenz : $\nabla \cdot f = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} \sigma(y-x) \\ rx - y - xz \\ xy - bz \end{bmatrix}$

$\rightsquigarrow \nabla \cdot \underset{\sim}{f} = -\sigma - 1 - b$. It happens to not be a function of (x, y, z). Const. everywhere.

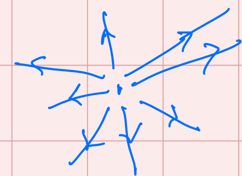
$\dot{V} = -(\sigma+1+b)V$

$V(t) = V(0)\, e^{-(\sigma+1+b)t}$

Because of dissipation

→ Quasiperiodicity is not allowed because q.p. flow occurs on the surface of a fixed torus in phase space. But if volumes in phase space are always shrinking, you can't have such an invariant torus.

→ No repelling fixed points or repelling closed orbits are allowed.



Fixed Points of Lorenz system

$(x,y,z) = \underline{0}$ is always a fixed pt.

$(x,y,z) = \left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1\right)$ (C^+, C^-) are fixed pts if $r > 1$

Linear stability of origin:

linearized Lorenz equations:

$$\dot{x} = \sigma(y-x)$$
$$\dot{y} = rx - y$$
$$\dot{z} = -bz$$

contracting direction

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma \\ r & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

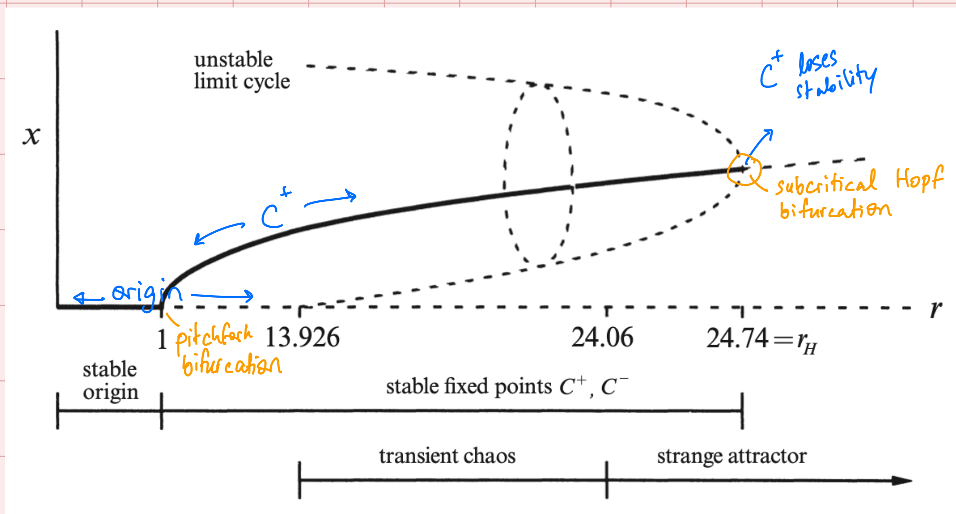
$\tau = -(\sigma+1)$

$\Delta = \sigma - \sigma r = \sigma(1-r)$

$r > 1$: saddle pt

$r < 1$: stable node

C^+ and C^- are stable for $1 < r < r_H = \dfrac{\sigma(\sigma+b+3)}{\sigma-b-1}$



so what exists after $r = r_H$?

- trajectories don't go out to infinity
- (it can be shown) there are no stable limit cycles
- no attracting fixed pts.
- but phase space is dissipative

<E91lorenz3>

⇝ A strange attractor

<u>Attractor</u> :

if "A" is an attractor, it has these properties →

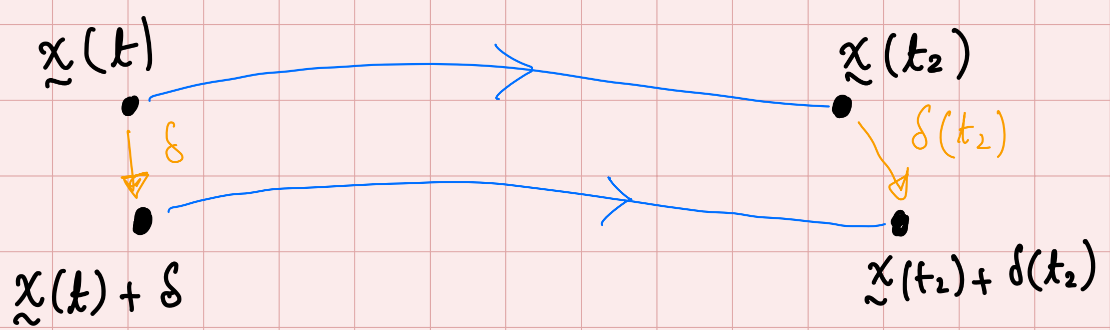
1. A is an *invariant set:* any trajectory $\mathbf{x}(t)$ that starts in A stays in A for all time.
2. *A attracts an open set of initial conditions*: there is an open set U containing A such that if $\mathbf{x}(0) \in U$, then the distance from $\mathbf{x}(t)$ to A tends to zero as $t \to \infty$. This means that A attracts all trajectories that start sufficiently close to it. The largest such U is called the *basin of attraction of A.*
3. A is *minimal:* there is no proper subset of A that satisfies conditions 1 and 2.

An attractor has a certain shape — what is the shape of the Lorenz attractor? — dimension "2.05"

tinyurl.com/E91lorenz3

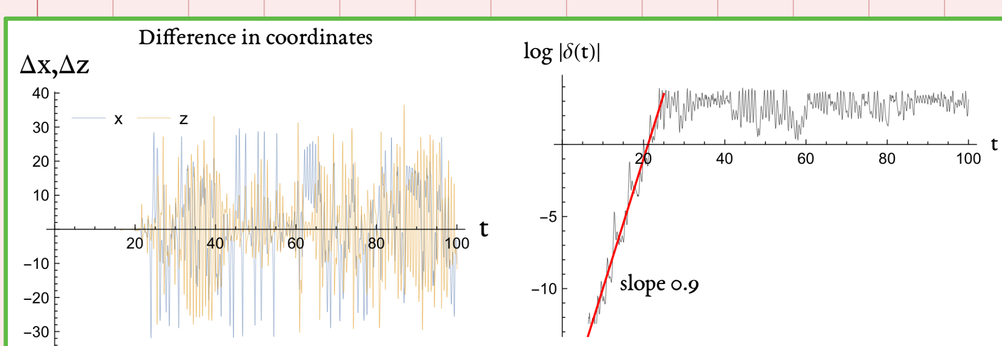
For Lorenz's parameters $\sigma = 10$, $b = 8/3$, $r = 28$, the system exhibits chaos. Strange attractors are chaotic ones.

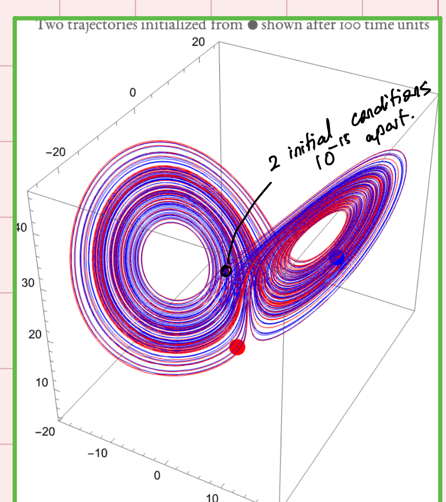
Strogatz: Chaos is aperiodic long-term behaviour in a deterministic system that shows sensitive dependence on initial conditions.



$\delta(t)$: how far apart are the two trajectories

use both as independent as initial conditions. → See how δ evolves





Mon, Apr 7 Lecture 19

Time Horizon of Prediction in systems with sensitive dependence on initial conditions.

$$\|\delta(t)\| \sim \|\delta_0\| e^{\lambda t}$$

for Lorenz system $\lambda \approx 0.9$ when $\sigma = 10, b = \frac{8}{3}, r = 28$

Find time t^* at which two initially nearby (δ_0) trajectories have diverged by more than ϵ.

$$\epsilon \approx \|\delta_0\| e^{\lambda t^*}$$

$$\Rightarrow \frac{\epsilon}{\|\delta_0\|} \approx e^{\lambda t^*} \Rightarrow \boxed{t^* \approx \frac{1}{\lambda} \log\left[\frac{\epsilon}{\|\delta_0\|}\right]}$$

λ: the largest Liapunov Exponent of the system. $= 0.9$ for Lorenz

Example

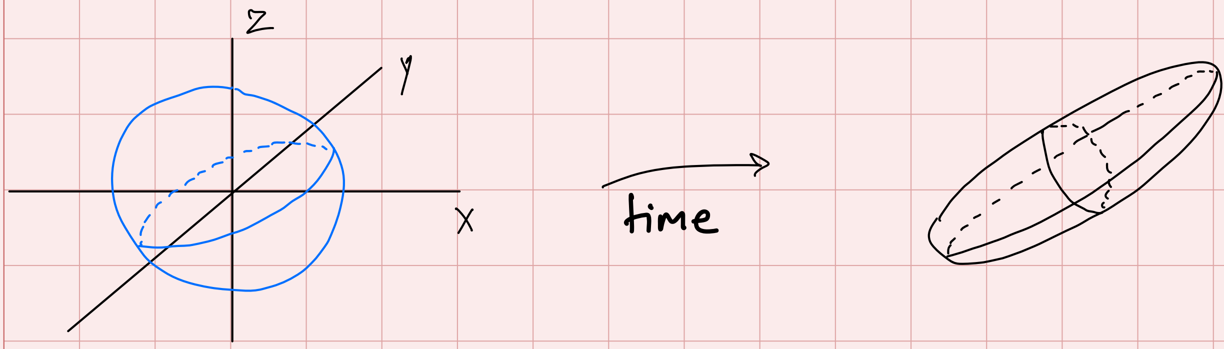
Two measurements were made to a precision of $\delta_0 = 10^{-7}$

We consider deviations more than $\epsilon = 10^{-3}$ to be unacceptable. $\Rightarrow t^* \approx 10.2$

Increase initial precision.

Now, $\delta_0 = 10^{-15}$ (100,000,000 × more precise) $\Rightarrow t^* \approx 30.7$ (3×)

What exactly is λ?



sphere of initial conditions with infinitesimal radius δ

Ellipsoid with principal axes

$$\delta_1(0) \longrightarrow \delta_1(0)e^{\lambda_1 t}$$
$$\delta_2(0) \longrightarrow \delta_2(0)e^{\lambda_2 t}$$
$$\delta_3(0) \longrightarrow \delta_3(0)e^{\lambda_3 t}$$

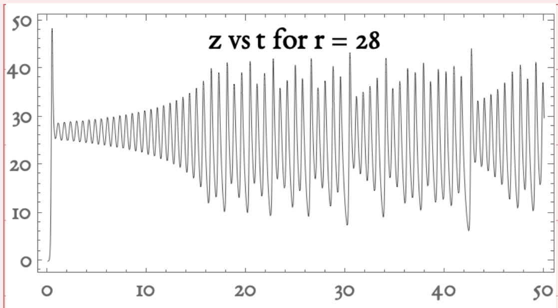
At large times, the largest λ_k controls the size of the ellipsoid.

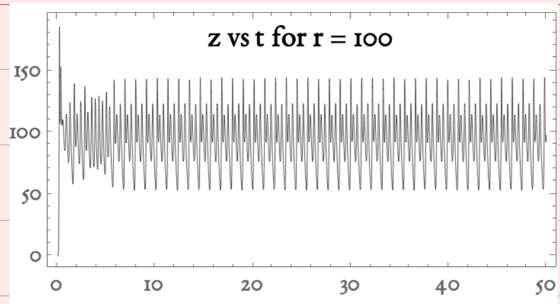
(largest λ_k: Liapunov Exponent.)

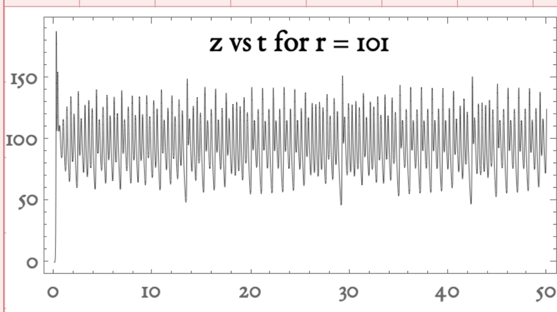
Negative $\lambda_k \Rightarrow$ that direction shrinks
Positive $\lambda_k \Rightarrow$ that direction expands.

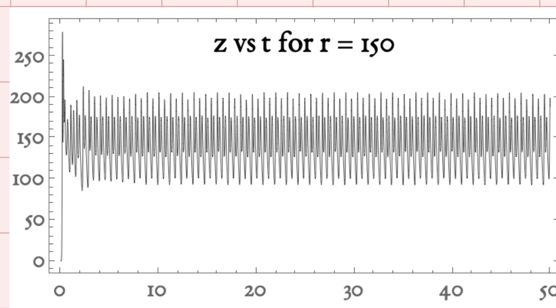
If Liapunov Exponent > 0, $\Rightarrow$ sensitive dependence on initial conditions.

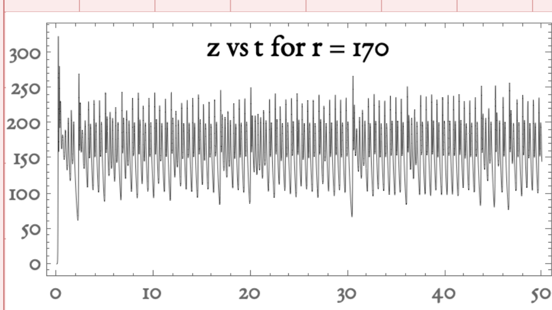
tinyurl.com/E91lorenz4 ⟶ remix

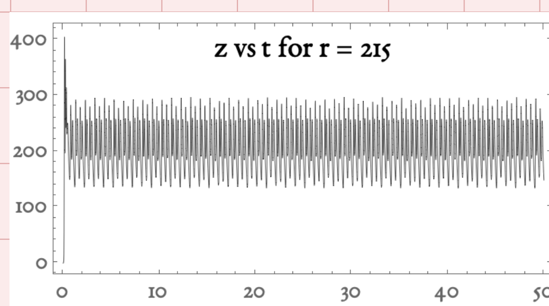












order → chaos → order → chaos → ?

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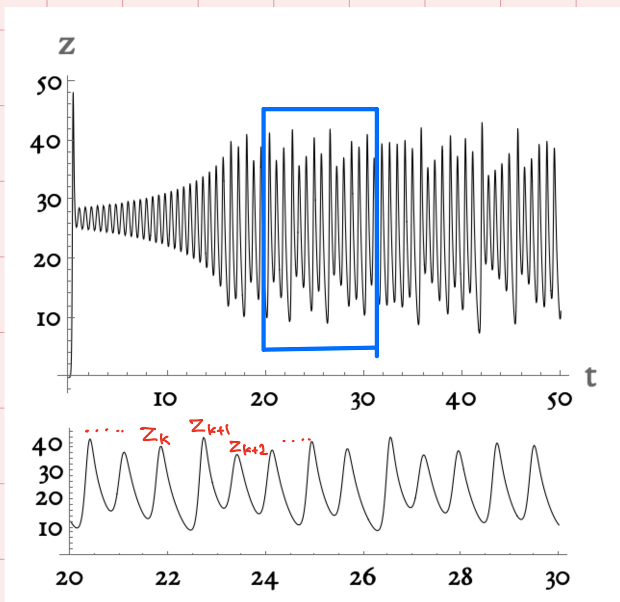


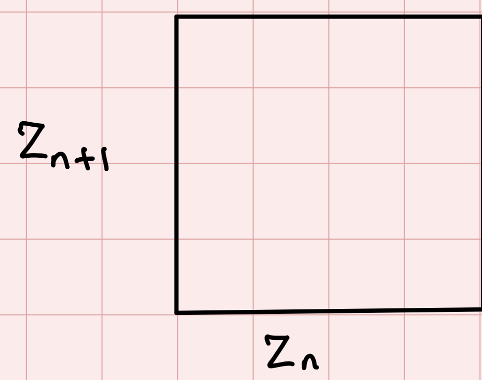
Continuous Time → Discrete Time

Differential Eqs → Maps

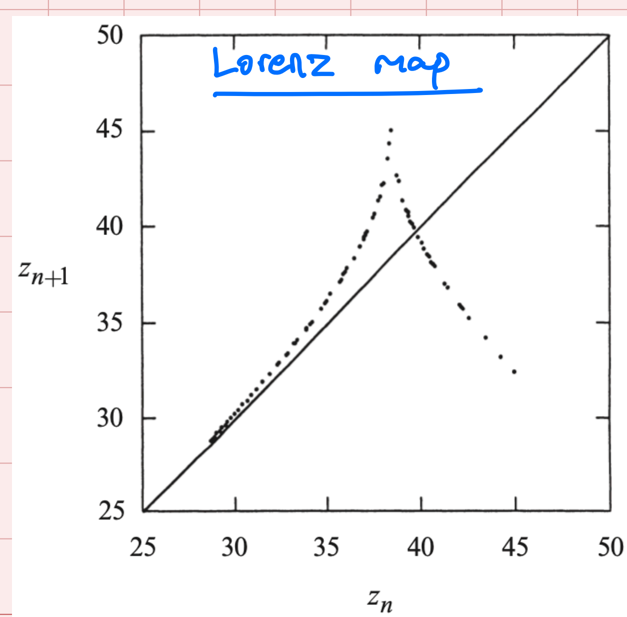
$\dot{x} = f(x)$ $x_{n+1} = f(x_n)$

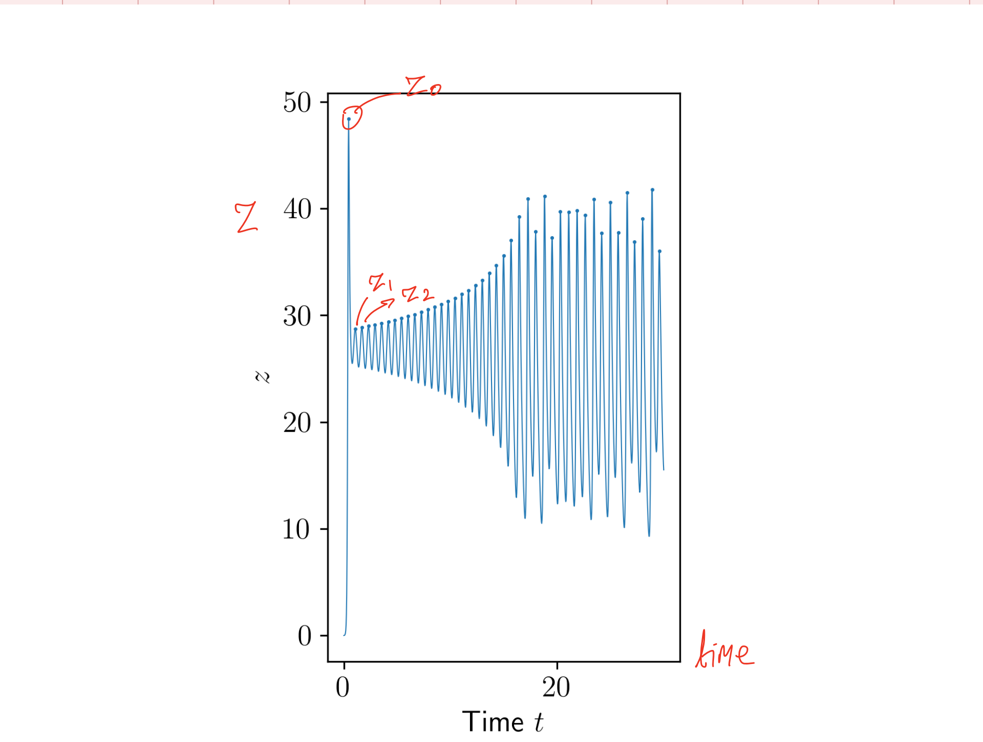
These two are not the same, even if you have a "map" and a differential eqn. describing the same underlying system.

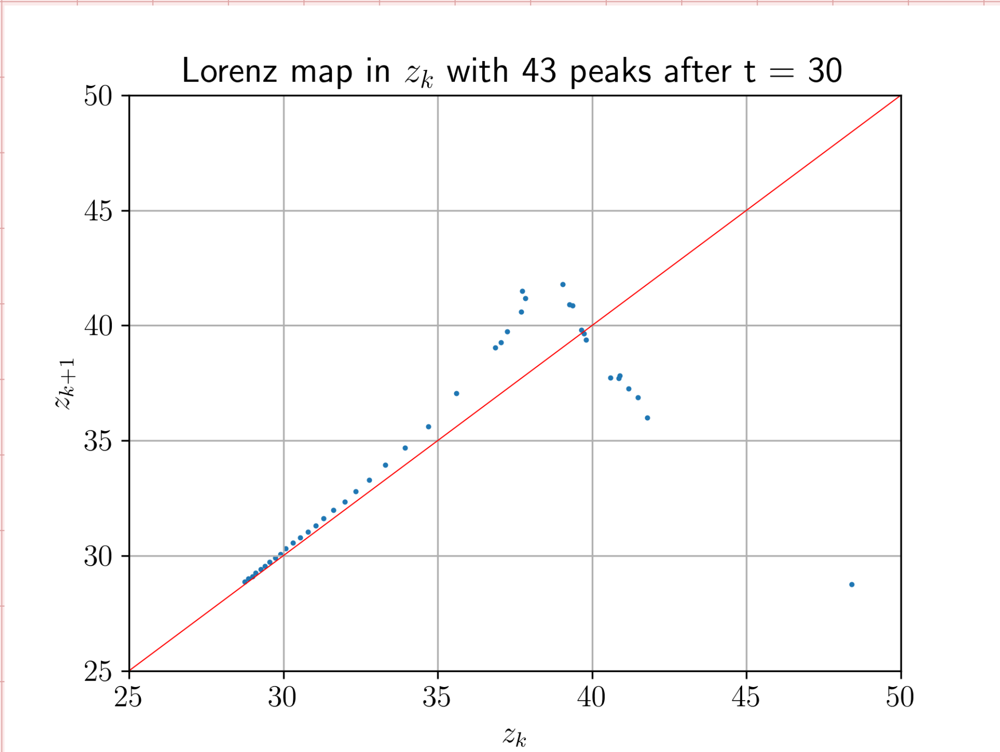
If you know "f" for a map, you can plot → by graphing the function.

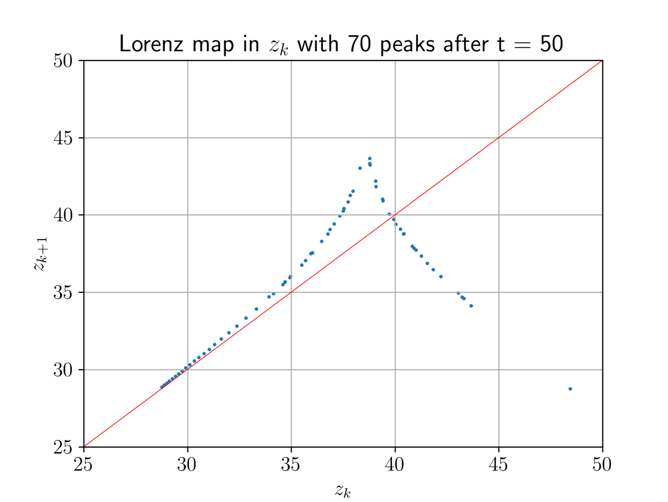


For Lorenz system, you have to do this empirically

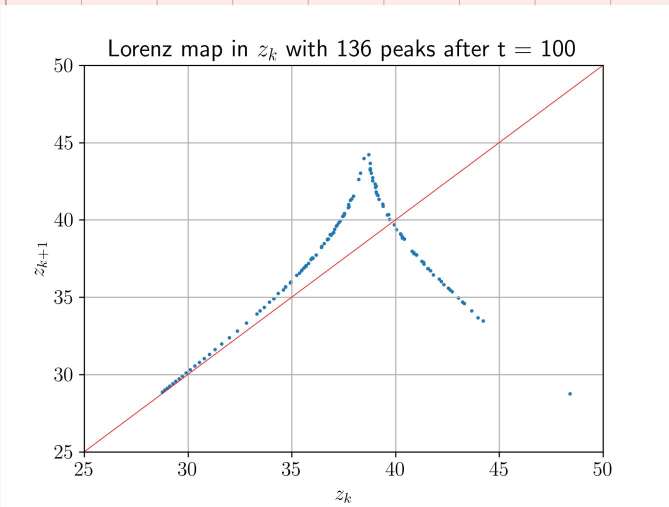



z0
z
z1
z2
z
time
Time t

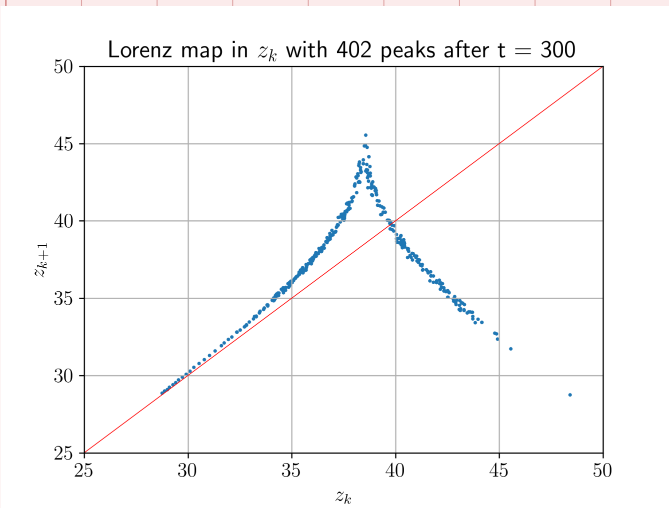

Lorenz map in z_k with 43 peaks after t = 30
z_{k+1}
z_k

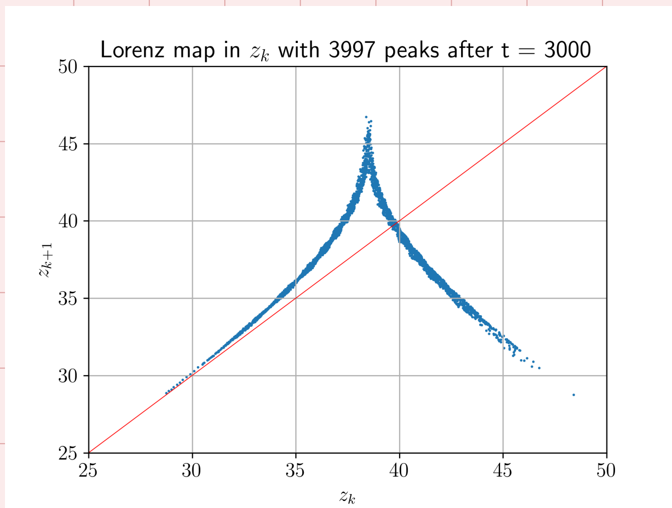


All figures produced with $\sigma = 10$, $b = 8/3$, $r = 28$ when Lorenz system is known to be chaotic.



looks like graph of $1 - \sqrt{|2z - 1|}$ with appropriate shifts and rescaling.





Why are we looking at maps?

→ Behaviour of Lorenz system is not well-understood. A simple(r) mathematical model that captures some aspects of the Lorenz system might give us some insight.

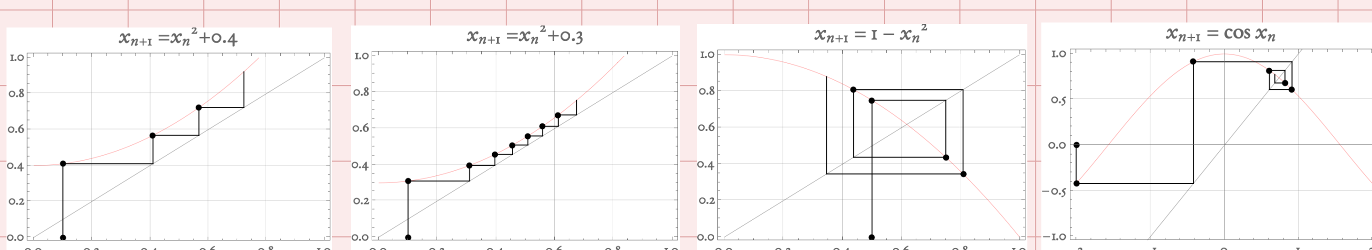
→ Can 1-d maps have chaos?

→ In some physical systems, it makes sense to think of time as discrete

$$x_{n+1} = f(x_n) \qquad f: \mathbb{R} \to \mathbb{R}$$

- f is a "map" from $\mathbb{R}$ to $\mathbb{R}$
- We'll study f's that are continuous and piecewise smooth. (no jumps; cusps/kinks are allowed)

Some more examples of maps

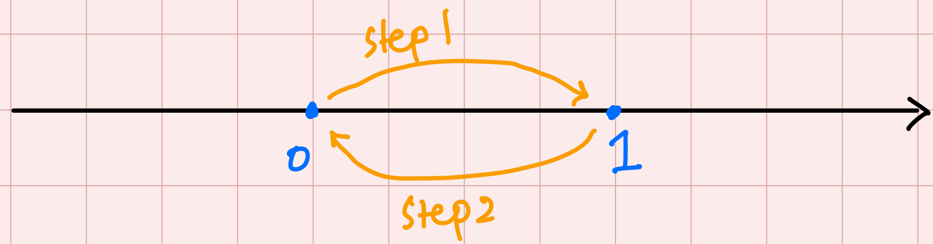


Mon, Apr 14 Lecture 21

Maps can have fixed points and periodic orb.

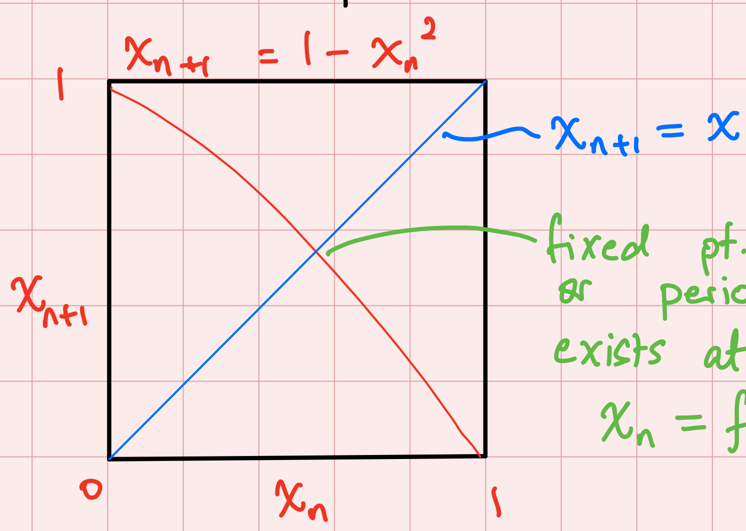
e.g. consider the map $x_{n+1} = 1 - x_n$ (*)

if $x_0 = 1$, $x_1 = 0$. $x_2 = 1$, $x_3 = 0$, ...



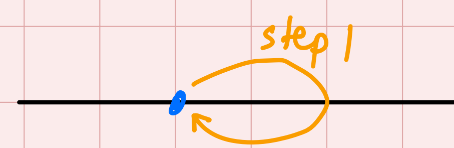
We have found a period − 2 orbit of the map (*)

Note: a fixed point can be called a period - 1 orbit.



fixed pt.
or period - 1 orbit.
exists at solution to eqn:

$x_n = f(x_n)$, or $x = 1 - x^2$



For a given f, how do we find period-n orbit?

$x = f(f(x))$ for $n = 2$

Stability on Maps

If $x^* = f(x^*)$ for some map $x_{n+1} = f(x_n)$

Let's see what happens to $x = x^* + \eta$, for small η, under the action of this map.

When $\eta = 0$, solution remains at x.

$$x_{n+1} = \underbrace{f(x^* + \eta_n)}_{} = \underbrace{f(x^*)}_{x^*} + \eta_n f'(x^*) + \underbrace{\frac{\eta_n^2}{2!} f''(\dots) + \dots}_{\text{ignore}}$$

What happens to $x^* + \eta$ under map?

$$\cancel{x^*} + \eta_{n+1} = \cancel{x^*} + f'(x^*)\,\eta_n$$

$$\boxed{\eta_{n+1} = \underbrace{f'(x^*)}_{\text{call this } \lambda.}\,\eta_n}$$

for any given fixed pt. x^*, $f'(x^*)$ is just a number.

and get a linearized map.

$$\eta_{n+1} = \lambda\,\eta_n$$

$$\boxed{\eta_n = \lambda^n \eta_0}$$

$\eta_1 = \lambda \eta_0$

$\eta_2 = \lambda \eta_1 = \lambda^2 \eta_0$

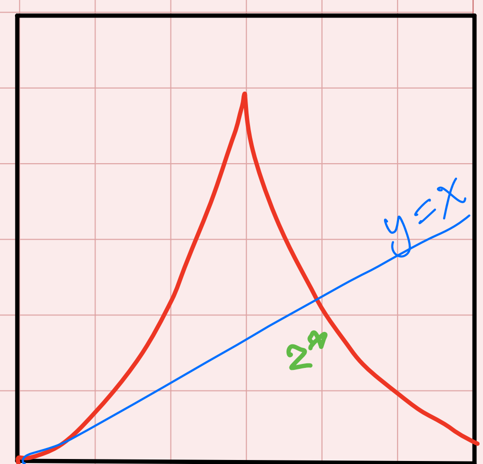
$\eta_3 = \lambda \eta_2 = \lambda^2 \eta_1 = \lambda^3 \eta_0$

if $|\lambda| < 1$, $\eta_n \to 0$ as $n \to \infty$

if $|\lambda| > 1$, $\eta_n \to \infty$ as $n \to \infty$

Evaluate Stability of fixed points in the Mathematical Model of the Lorenz map.

$$z_{n+1} = 1 - |2z_n - 1|^{1/2}$$



- where is the fixed pt?
- is it stable?

$$x = 1 - |2x-1|^{1/2}$$

$$\sqrt{|2x-1|} = 1-x$$

$$2x-1 = (1-x)^2$$

$$\Rightarrow \quad x^2 - 4x + 2 = 0$$

$$x = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \frac{\sqrt{8}}{2} = 2 \pm \sqrt{2}$$

$$\boxed{z^* = 2 - \sqrt{2}}$$

$f'(x) = ?$ $\quad f(x) = 1 - \sqrt{2x-1}$, ignore the $|\cdot|$

$$f'(x) = -\frac{1}{2}(2x-1)^{-1/2} \cdot 2$$

$$= -\frac{1}{\sqrt{2x-1}}$$

Now, evaluate at $x = x^*$

$$f'(x^*) = -\frac{1}{\sqrt{4-2\sqrt{2}-1}} = -\frac{1}{\sqrt{3-2\sqrt{2}}} = -1-\sqrt{2} \approx -2.41$$

$f'(x^*) \approx -2.41 \Rightarrow$ unstable.

The Logistic Map

The Logistic Map $\longrightarrow x_{n+1} = r\, x_n (1 - x_n)$

$r \to$ parameter.

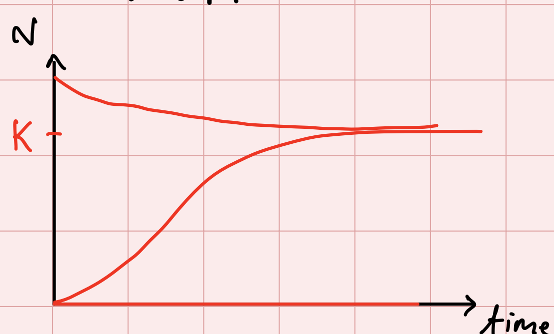
$0 < r \leq 4$

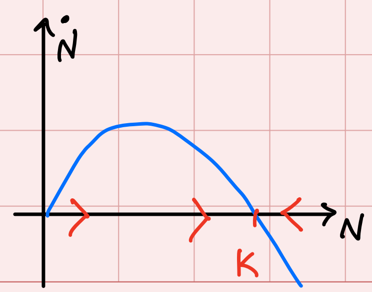
recall: $\dot{N} = rN\left[1 - \frac{N}{K}\right]$

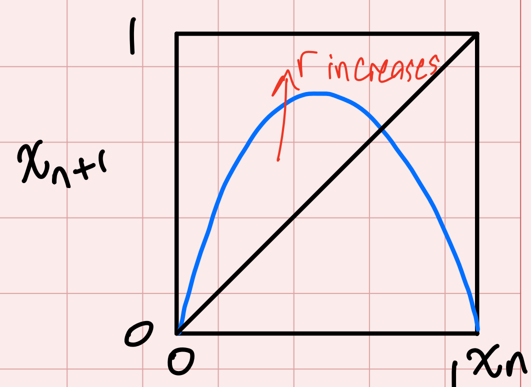
carrying capacity K

growth rate r

This system has the following behaviour:

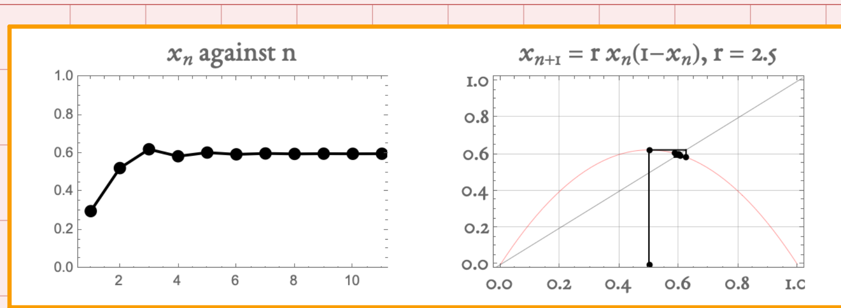




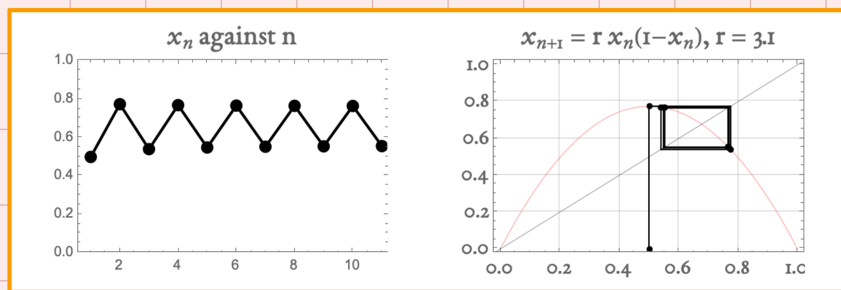


We now observe the logistic map's behaviour for different values of r

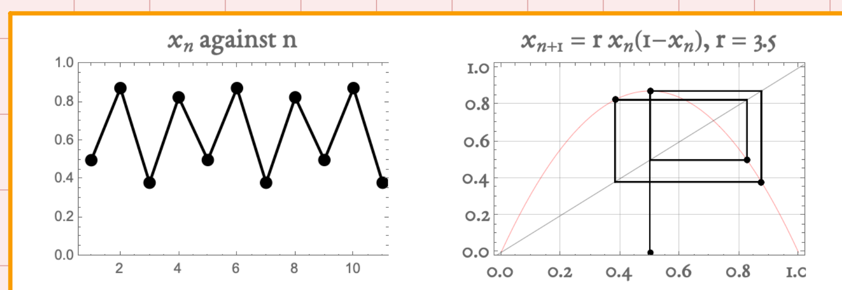
$r = 2.5$



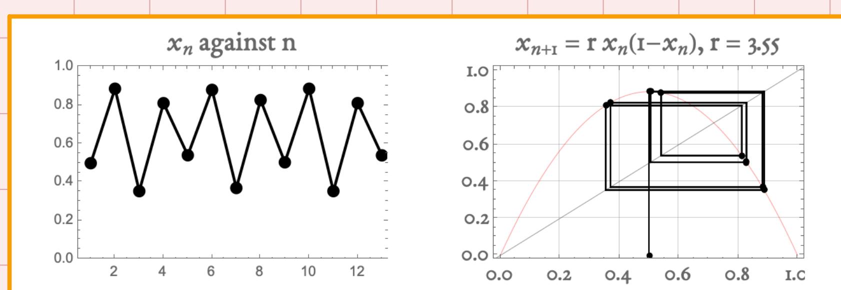
$r = 3.1$



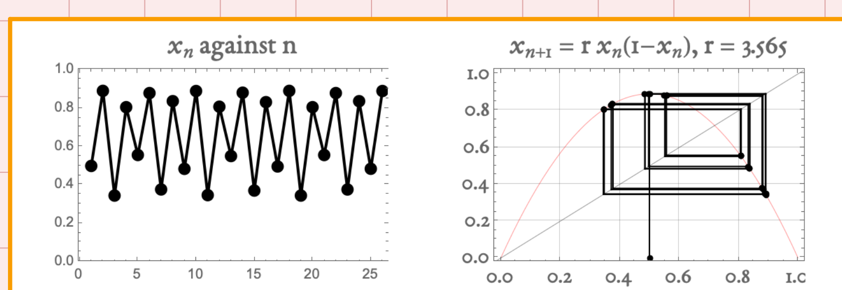
$r = 3.5$



$r = 3.55$



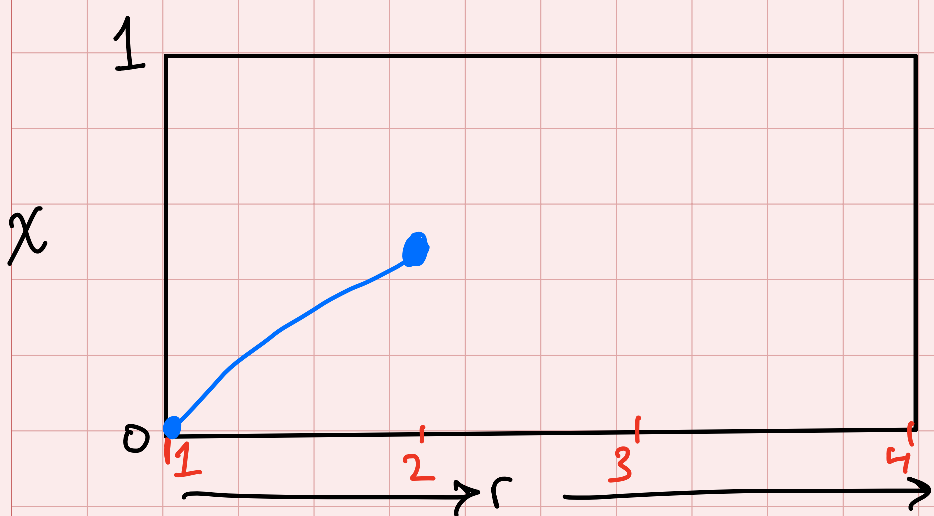
$r = 3.565$



Wed, Apr 16 Lecture 22

As r increases, the period of the cycle doubles. (Successively)

⇒ Draw the orbit diagram (of the Logistic map)



similar to a bifurcation diagram but only includes stable structures. → orbits: period-1 orbits, period-2 orbits, ... (fixed pts)

At $r = 1$, how does logistic map behave?

Look for solutions to $\{ x_{n+1} = x_n(1-x_n)\cdot r \ , \ x_n = x_{n+1} \}$

$$x = x(1-x)$$

$$x = x - x^2 \Rightarrow x = 0 \text{ only solution.}$$

At $r=2$: Solve $x = 2x(1-x)$ ($r=2$)

$$x = 2x - 2x^2$$
$$2x^2 - x = 0$$
$$x(x - 1/2) = 0$$

$x = 1/2$ happens to be stable

two solutions! (ignore $x=0$; unstable)

At $r = 3.5$: ...　　$x_{n+1} = \frac{7}{2} x_n (1 - x_n)$, $x_{n+1} = x_n$

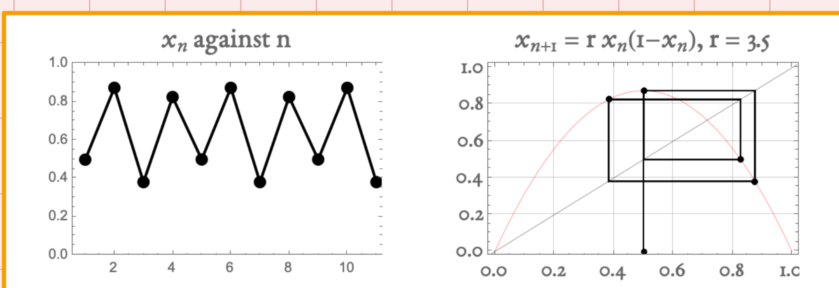
⋮

$$x(x - 5/7) = 0$$

$x = 5/7$ is a fixed pt.

↓

unstable, doesn't make it into the orbit diagram.



Fixed points of the 2^{nd} iterate map correspond to period-2 orbits.

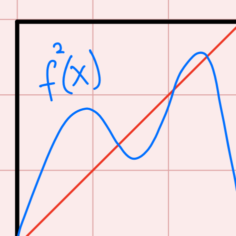
Example

i.e find x_n such that $x_{n+2} = x_n$　　for some map f.

$$f(f(x_n)) = x_n$$

$\Rightarrow$ a period-k orbit is solution of

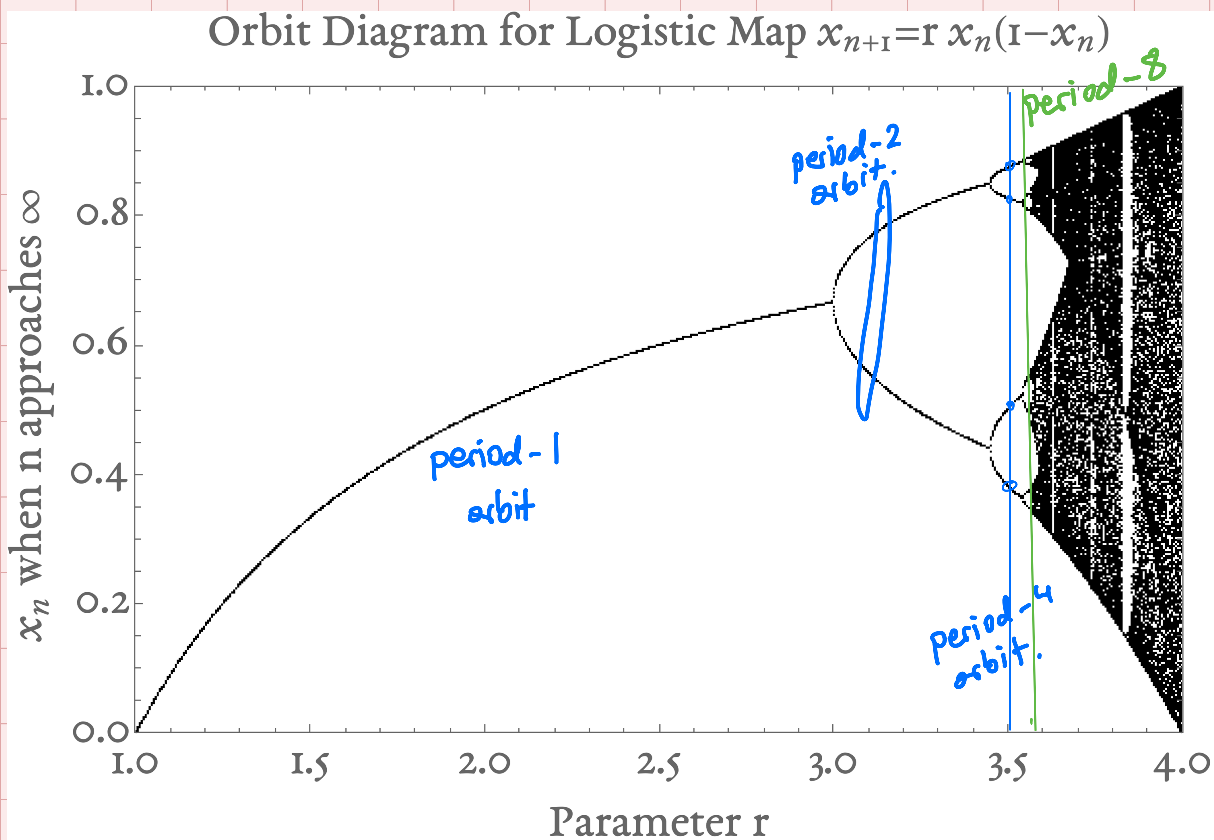
$$x_{n+k} = x_n, \quad f^k(x) = x.$$



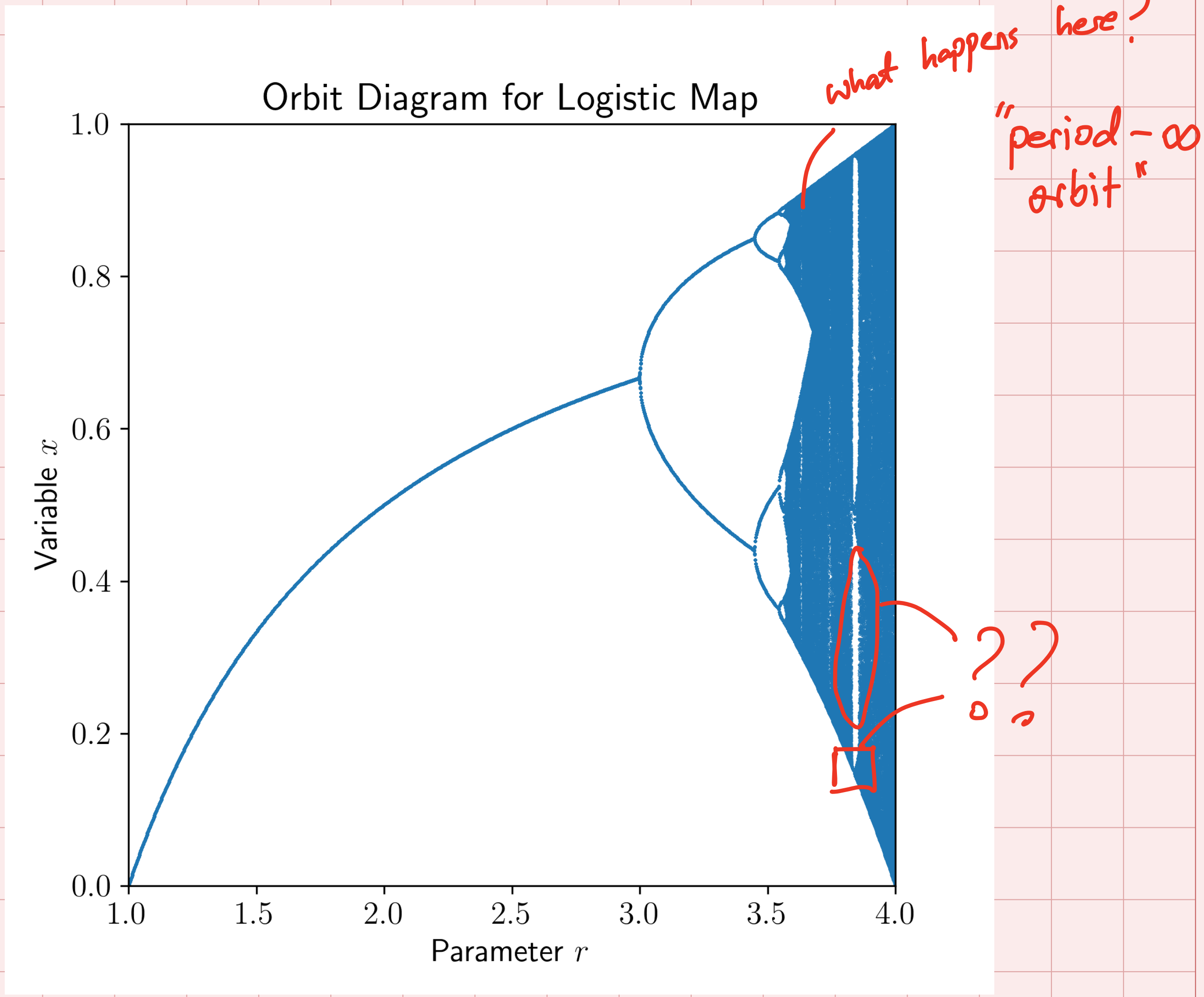
Typically, graphical methods are used.

Period-doubling route to chaos

At large r, between ~3.7 and 4, the logistic map shows signs of chaos.



- Pick r.
- we start out with any value of x
- Iterate the map, say, 1200 times. $x_0 \rightarrow x_1 \rightarrow x_2$
- Drop the first, say, 1000 values.
- Plot all of the most recent 200 values.

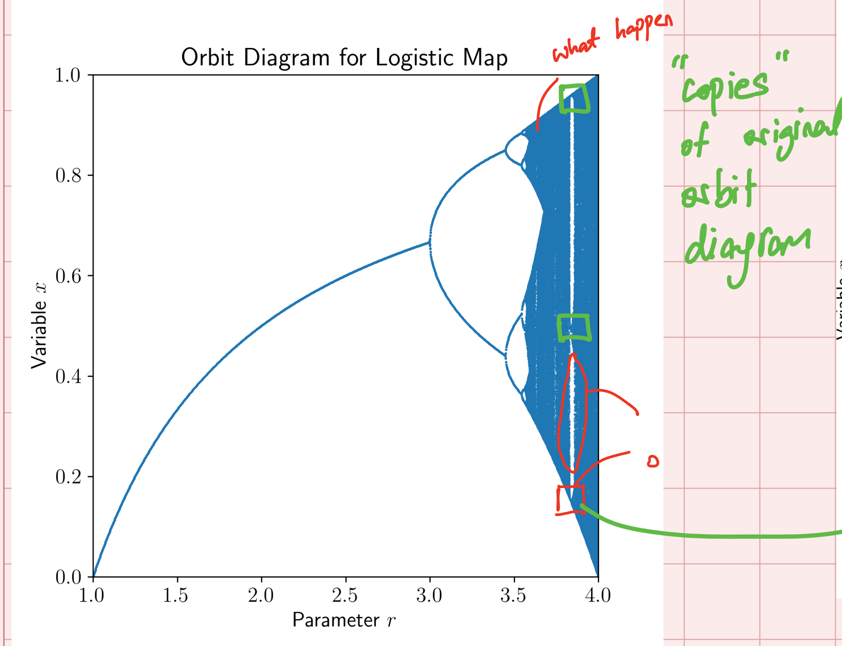


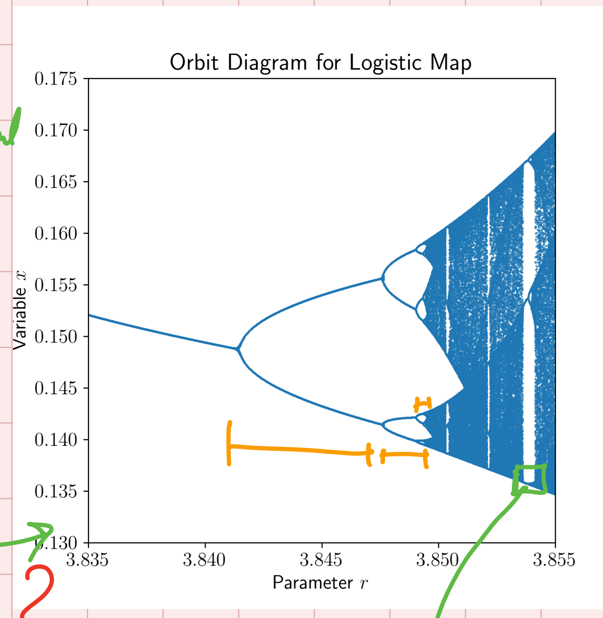
- A period-k orbit comes back after k steps.
- A period-∞ orbit comes back after ∞ steps.

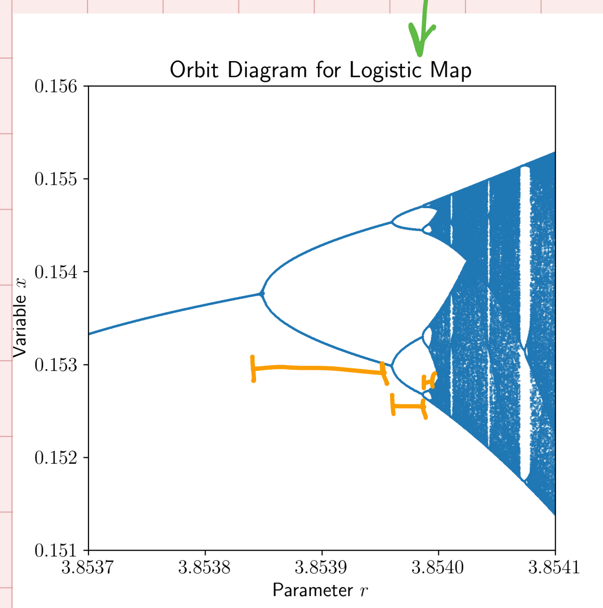
never repeats itself exactly

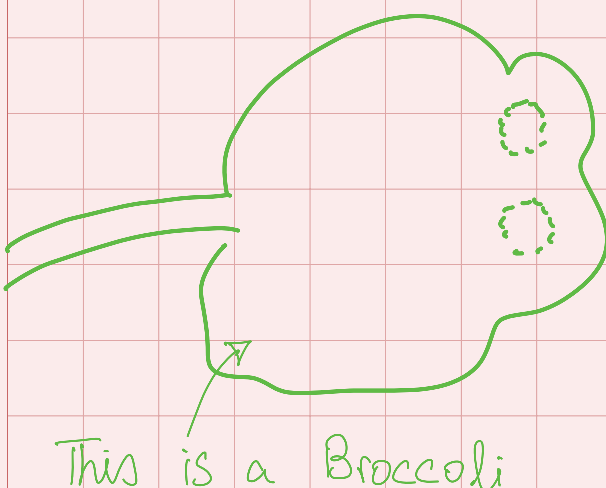
At some limiting value of r, k ⟶ ∞

a finite irrational number


Orbit Diagram for Logistic Map
Variable x
Parameter r
1.0 1.5 2.0 2.5 3.0 3.5 4.0
0.0 0.2 0.4 0.6 0.8 1.0
what happen
"copies" of original orbit diagram

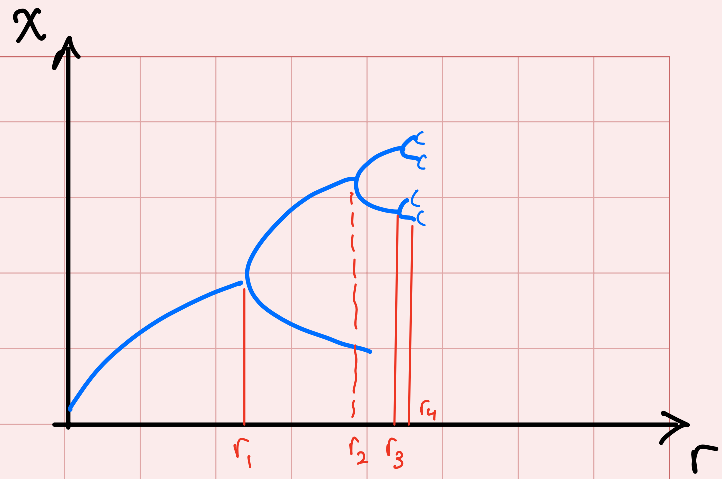

Orbit Diagram for Logistic Map
Variable x
Parameter r
3.835 3.840 3.845 3.850 3.855
0.130 0.135 0.140 0.145 0.150 0.155 0.160 0.165 0.170 0.175


Orbit Diagram for Logistic Map
Variable x
Parameter r
3.8537 3.8538 3.8539 3.8540 3.8541
0.151 0.152 0.153 0.154 0.155 0.156


This is a Broccoli

Mon, Apr 21 Lecture 23

Features of Orbit Diagram

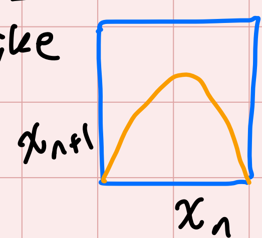


$$\delta = \lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} \approx 4.669$$

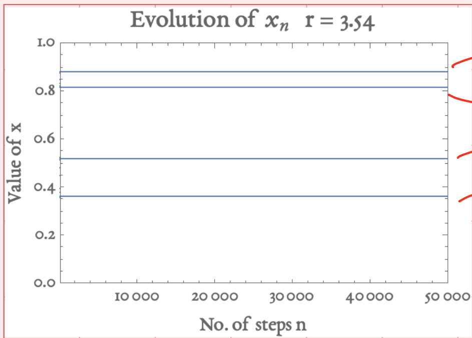
Feigenbaum's constant δ

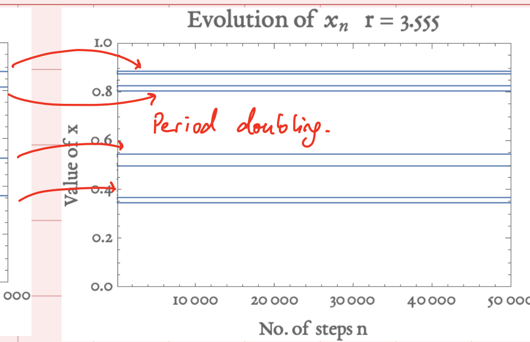
characterizes period-doubling in all unimodal maps

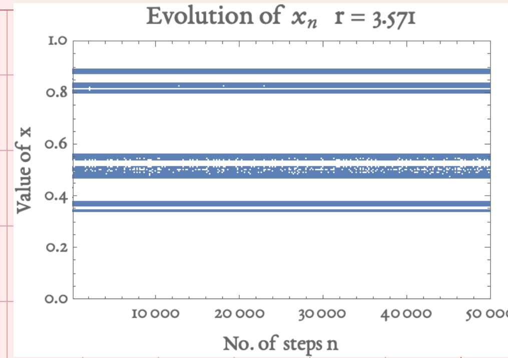
i.e. anything with a graph like

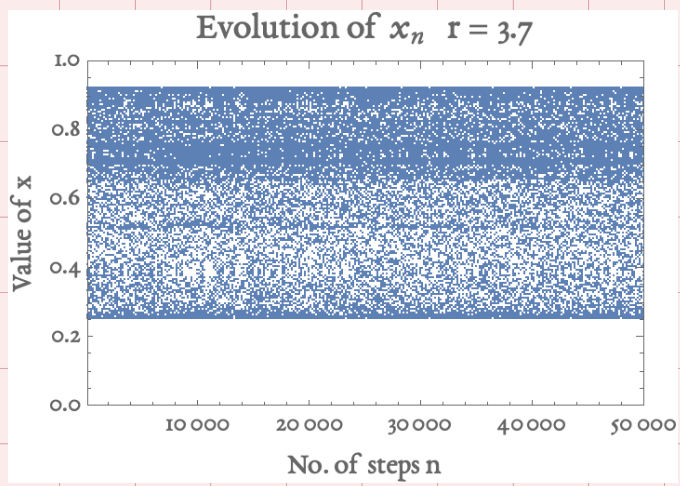


For some value of r, $3.55 < r < 3.70$, the system becomes chaotic : orbit diagram is dense: no value is ever repeated, but you basically cover "all numbers" b/w 0 and 1

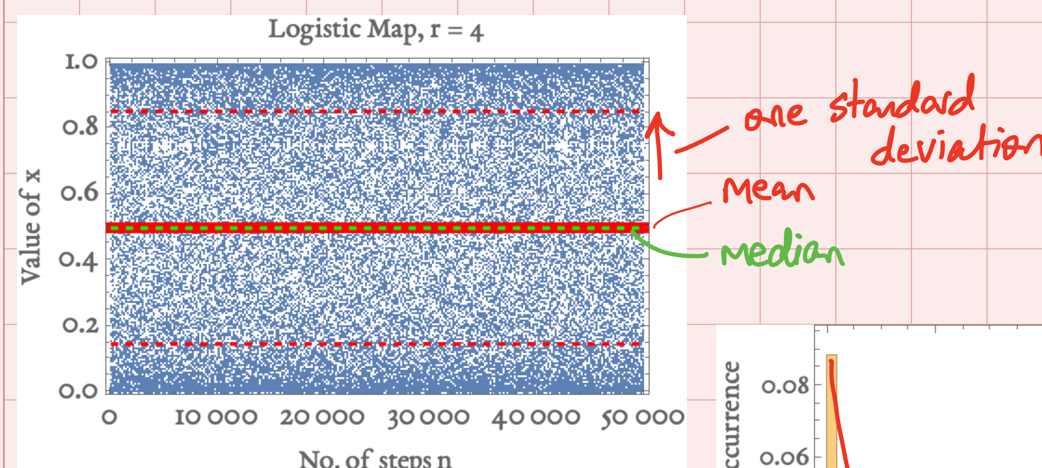

Evolution of x_n r = 3.54
Value of x
No. of steps n

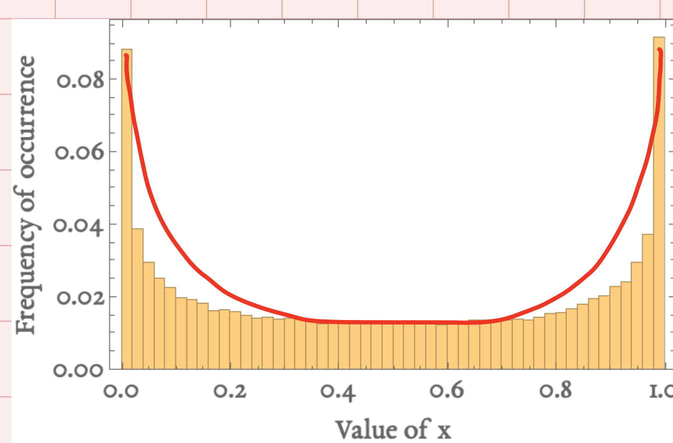

Evolution of x_n r = 3.555
Period doubling.
Value of x
No. of steps n


Evolution of x_n r = 3.571
Value of x
No. of steps n


Evolution of x_n r = 3.7
Value of x
No. of steps n

$$x_{n+1} = r x_n (1 - x_n)$$


Logistic Map, r = 4
Value of x
No. of steps n
one standard deviation
mean
median


Frequency of occurrence
Value of x

[PDF]

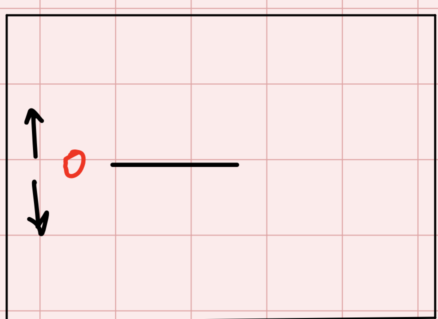
The probability density function that emerges from this data — shaped like ∪ — is reminiscent of the probability distribution known as arcsine:

A distribution whose cumulative distribution (CDF) is $\frac{2}{\pi}\arcsin\sqrt{x}$ and its PDF is $\frac{1}{\pi\sqrt{x(1-x)}}$

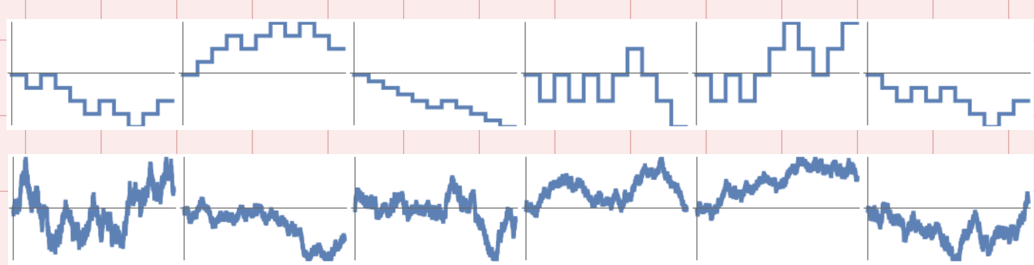
~ related to random walks

each step: two possibilities: up (+1), down (−1) } take n steps.

probability 1/2



over time, is it more likely that he will spend more time on one side of the line or is it more likely that he will spend equal amounts of time above & below?

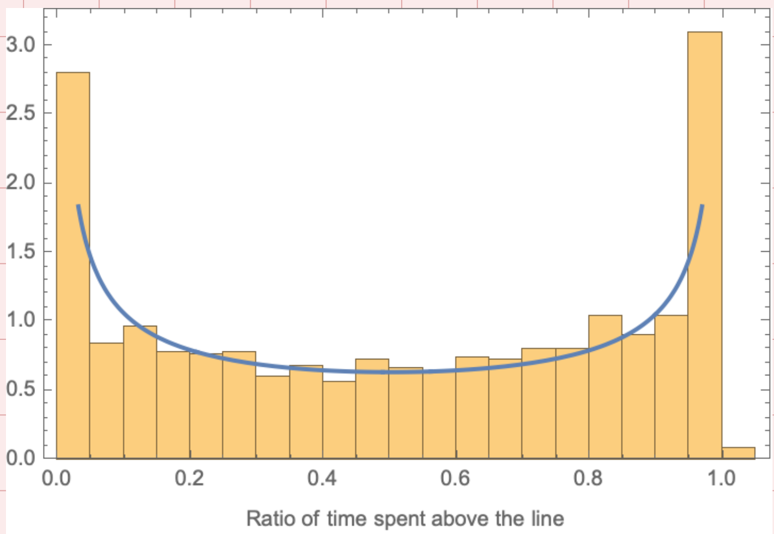


10 steps

10,000 steps

cumulative area under curve is the probability that one person will win a long coin-toss game. If area is zero, a tie will occur. ⇒ Ties are unlikely.

This process leads to an arcsine distribution of outcomes



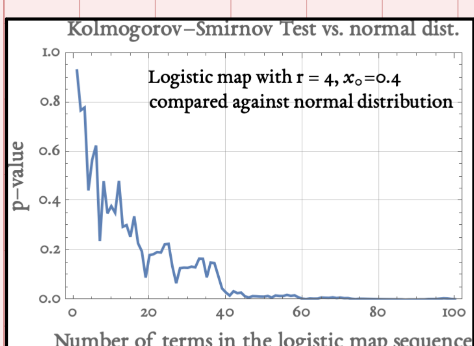
When a large number of coin-toss games are sampled, the proportion of 'time' for which one player is in the lead is very high, and a long-running tie is unlikely.

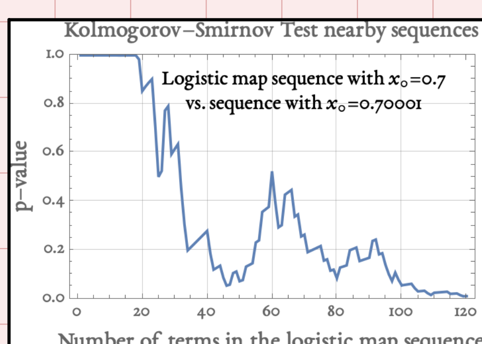
Player 1 winning by large margin — Tie — Player 2 winning by large margin

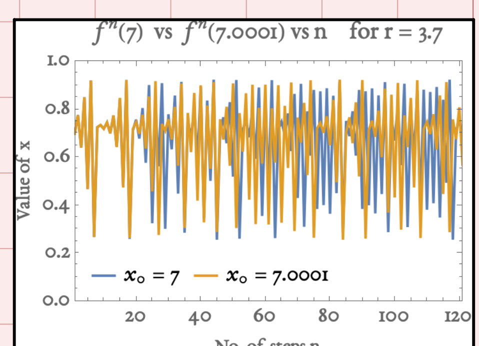
Kolmogorov-Smirnov Test

Compare a sequence $\{x_0\ x_1\ x_2 \ldots x_n\}$ against:-

1) a normal distribution
2) a different sequence with slightly different x_0.







p-value: probability that data came from tested distribution

Autocorrelation Test

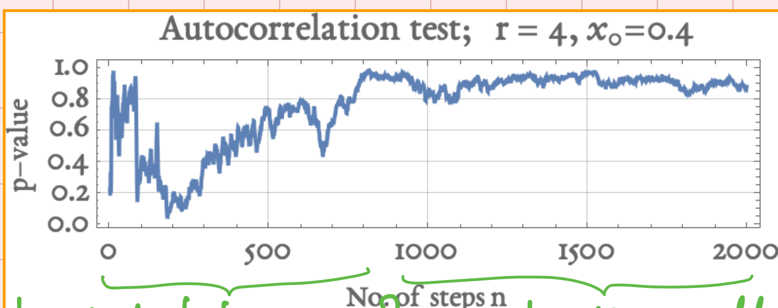
Is a signal correlated with itself? Noise is not correlated with anything, not even itself.

$\{a, b, c, d, \ldots\ldots, x, y, z, \ldots\}$

shift left.

$\{c, d, e, \ldots\ldots, x, y, z, \ldots\ldots\}$

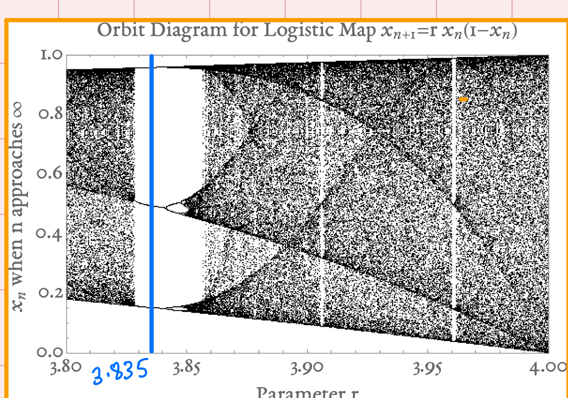
compare: are they close? if yes, $\Rightarrow$ autocorrelated.



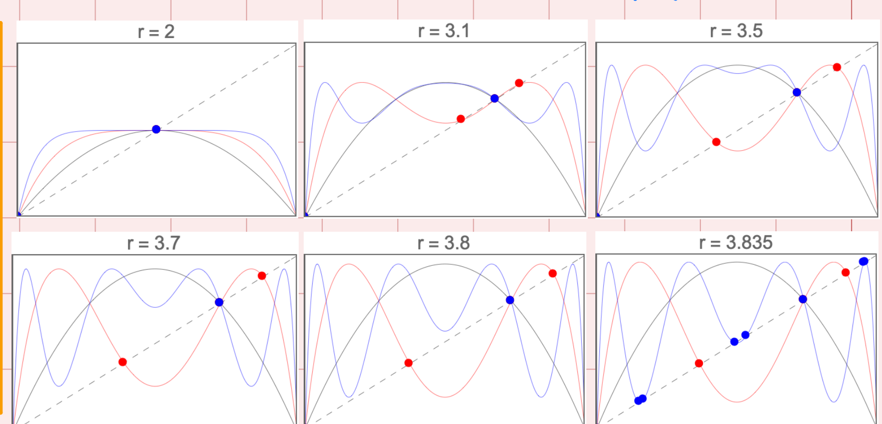
transient behaviour? chaotic attractor?

Existence of "periodic windows" can be predicted by examining the n^{th} iterate map $f^n(x)$

— : $f(x)$ — $f^2(x)$ — $f^3(x)$



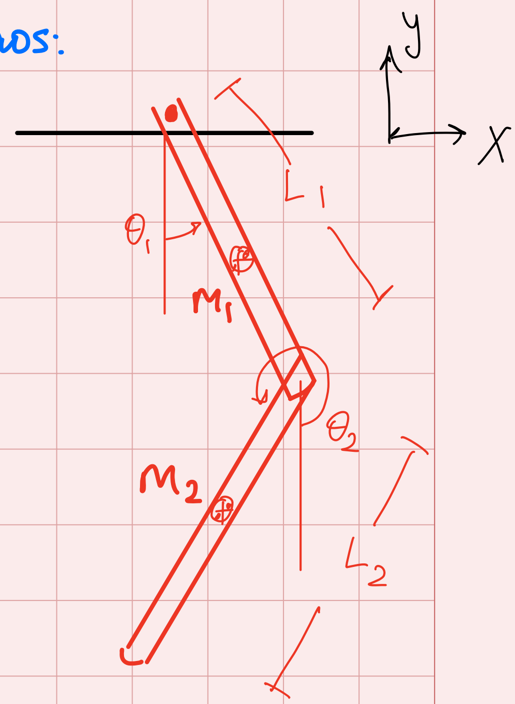
period-3 orbit



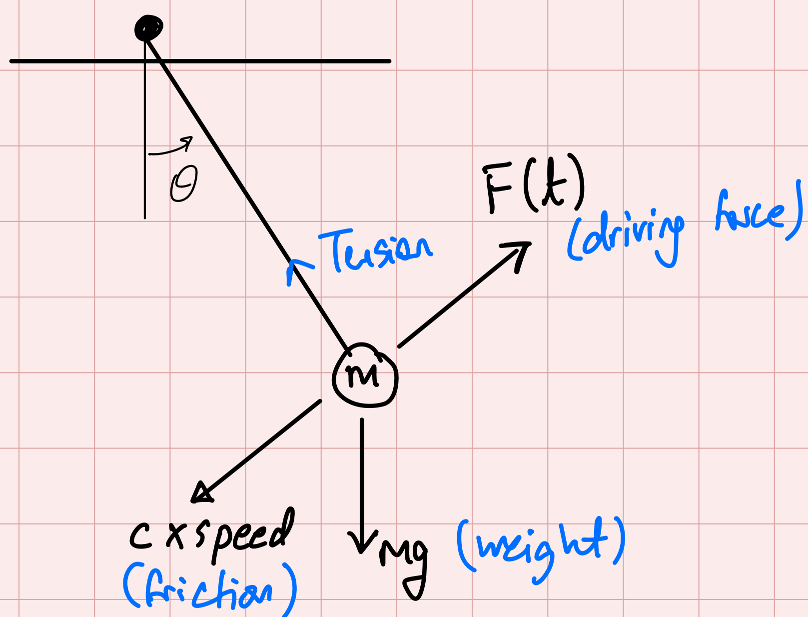
Wed, Apr 23 Lecture 24

Physical Systems that exhibit Chaos:

1) The double (compound) pendulum
distributed mass
with or without damping.



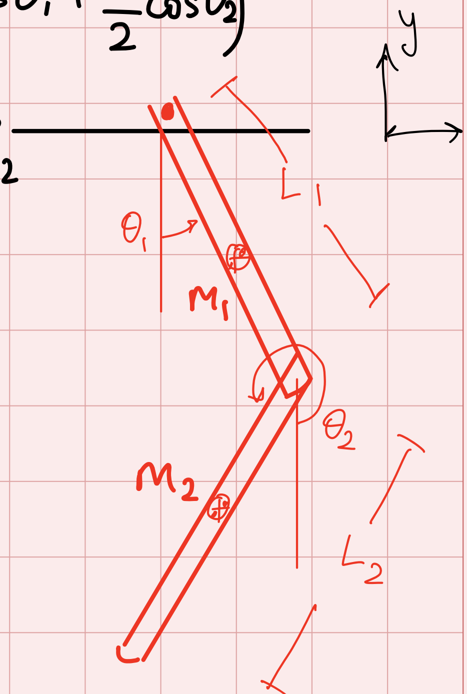
2) The simple driven pendulum
(with damping)



① Potential Energy $V = m g y_1 + m g y_2$

$$= -m_1 g \frac{L_1}{2}\cos\theta_1 - m_2 g\left(L_1\cos\theta_1 + \frac{L_2}{2}\cos\theta_2\right)$$

Kinetic Energy $T = \frac{1}{2} m_1 \vec{v}_1 \cdot \vec{v}_1 + \frac{1}{2} m_2 \vec{v}_2 \cdot \vec{v}_2$



$$\vec{r}_1 = \begin{bmatrix} +\frac{L_1}{2}\sin\theta_1 \\ -\frac{L_1}{2}\cos\theta_1 \end{bmatrix}$$

$$\vec{r}_2 = \begin{bmatrix} +L_1\sin\theta_1 + \frac{L_2}{2}\sin\theta_2 \\ -L_1\cos\theta_1 - \frac{L_2}{2}\cos\theta_2 \end{bmatrix}$$

$$\vec{v} = \frac{d}{dt}(\vec{r}) \Rightarrow \vec{v}_1 = \begin{bmatrix} \frac{L_1}{2}\dot{\theta}_1\cos\theta_1 \\ \frac{L_1}{2}\dot{\theta}_1\sin\theta_1 \end{bmatrix}$$

similarly for $\vec{v}_2$.

$$\therefore \quad T = \underbrace{\frac{L_1^2 m_1}{8}\dot{\theta}_1^2}_{\text{first rod}} + \underbrace{\frac{1}{2} m_2 L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_1 L_1 L_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + \frac{1}{8} m_2 L_2^2 \dot{\theta}_2^2}_{2^{nd}\text{ rod}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_j}\right) - \frac{\partial L}{\partial \theta_j} = \begin{Bmatrix} \text{non-conservative} \\ \text{force terms} \end{Bmatrix}$$

e.g. "$-c\dot{\theta}$"

$j = \{1, 2\}$

Two nonlinear 2^{nd} order non-autonomous differential eq.
- each equation can have $\{\ddot{\theta}_1\ \dot{\theta}_1\ \theta_1\ \ddot{\theta}_2\ \dot{\theta}_2\ \theta_2\}$
- coupled — can't solve separately.
- can convert to four 1^{st} order ODEs

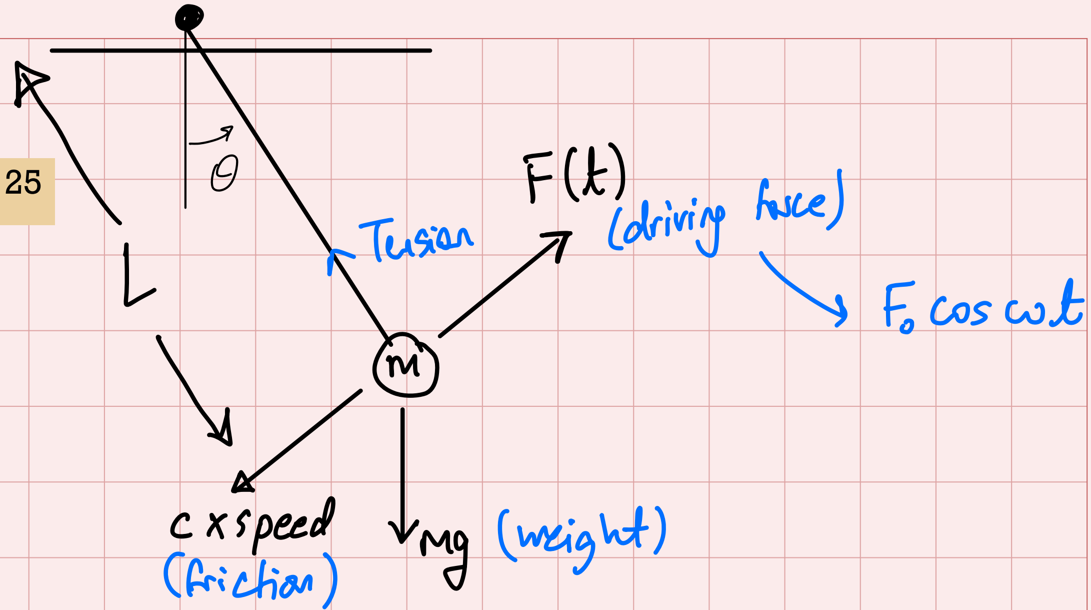
unknowns: $\begin{Bmatrix} \dot{\theta}_1(t) & \dot{\theta}_2(t) \\ \theta_1(t) & \theta_2(t) \end{Bmatrix} \longrightarrow$ call them x_i

$$\dot{\vec{x}} = \vec{f}(\vec{x})$$

$$\begin{aligned} \dot{x}_1 &= f_1(x_1\ x_2\ x_3\ x_4) \\ \dot{x}_2 &= f_2(x_1\ x_2\ x_3\ x_4) \\ \dot{x}_3 &= f_3(x_1\ x_2\ x_3\ x_4) \\ \dot{x}_4 &= f_4(x_1\ x_2\ x_3\ x_4) \end{aligned}$$

- Phase space is 4-dimensional.
- if you use small-angle approximation (carefully) then $\vec{f}(\vec{x})$ becomes $A\vec{x}$

Mon, Apr 28 Lecture 25



$$\underbrace{I}\,\ddot{\theta} = \sum \text{torques}$$

2nd order nonautonomous

$$mL^2\ddot{\theta} = -cL^2\dot{\theta} - mgL\sin\theta + L\cdot F(t)$$

$$\ddot{\theta} + \underbrace{\frac{c}{M}}_{2\beta}\dot{\theta} + \underbrace{\frac{g}{L}}_{\omega_0^2}\sin\theta = \underbrace{\frac{F_0}{ML}}_{\gamma\omega_0^2}\cos\omega t$$

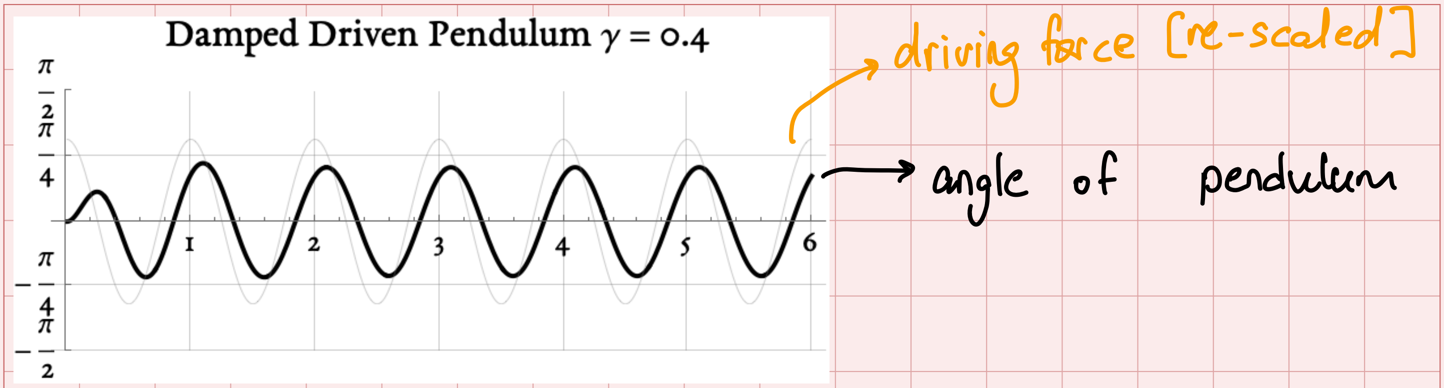
$$\boxed{\ddot{\theta} + 2\beta\dot{\theta} + \omega_0^2\sin\theta = \gamma\omega_0^2\cos\omega t}$$

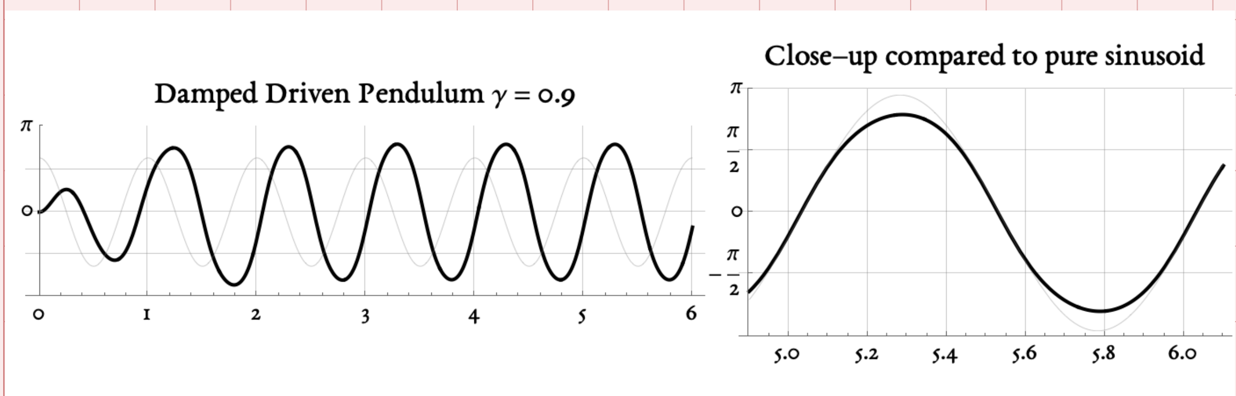
β : damping parameter [1/sec]

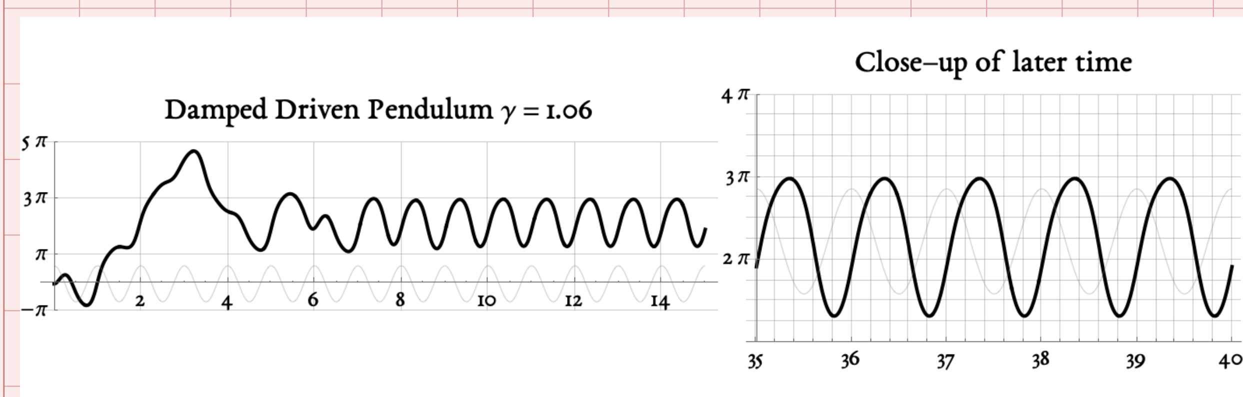
ω_0 : natural frequency [1/sec]

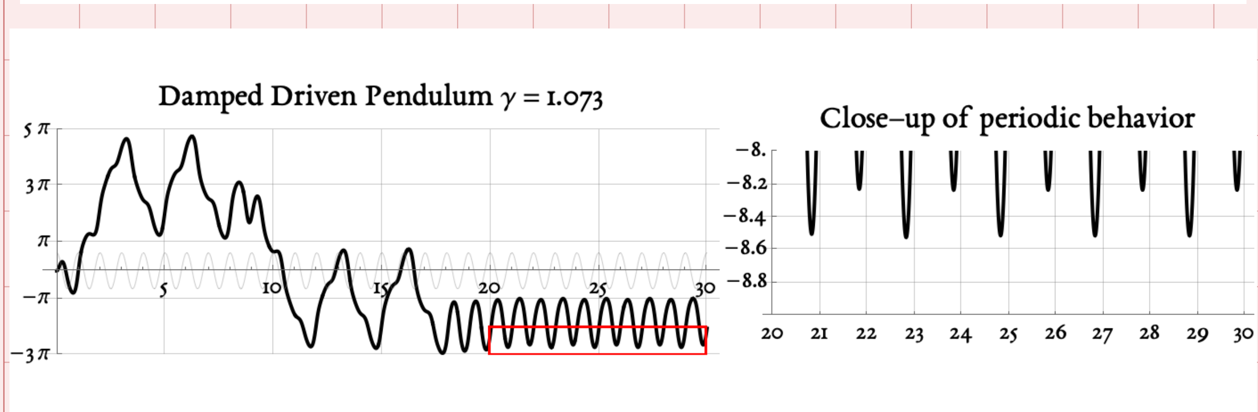
γ : drive strength [–] = F_0/mg

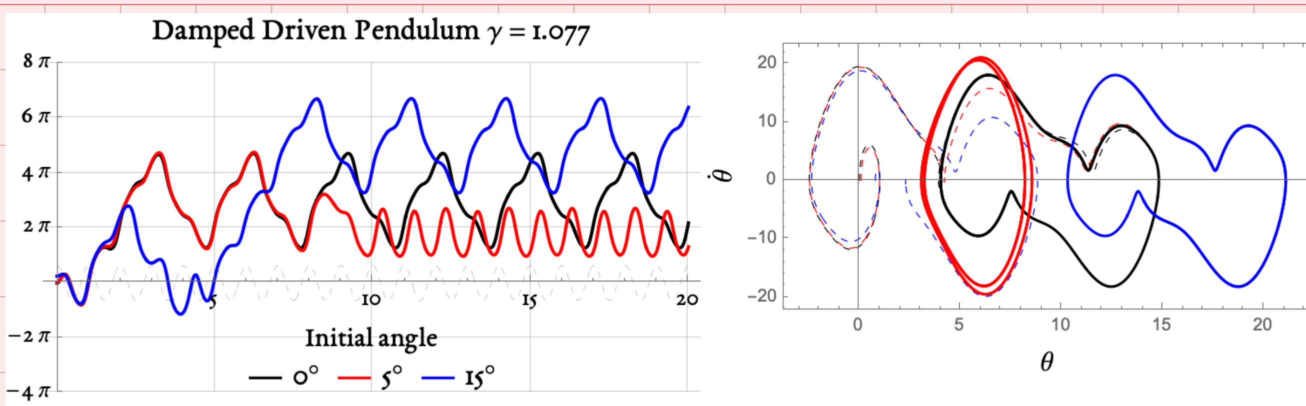
γ : tuning parameter


Damped Driven Pendulum γ = 0.4
driving force [re-scaled]
angle of pendulum

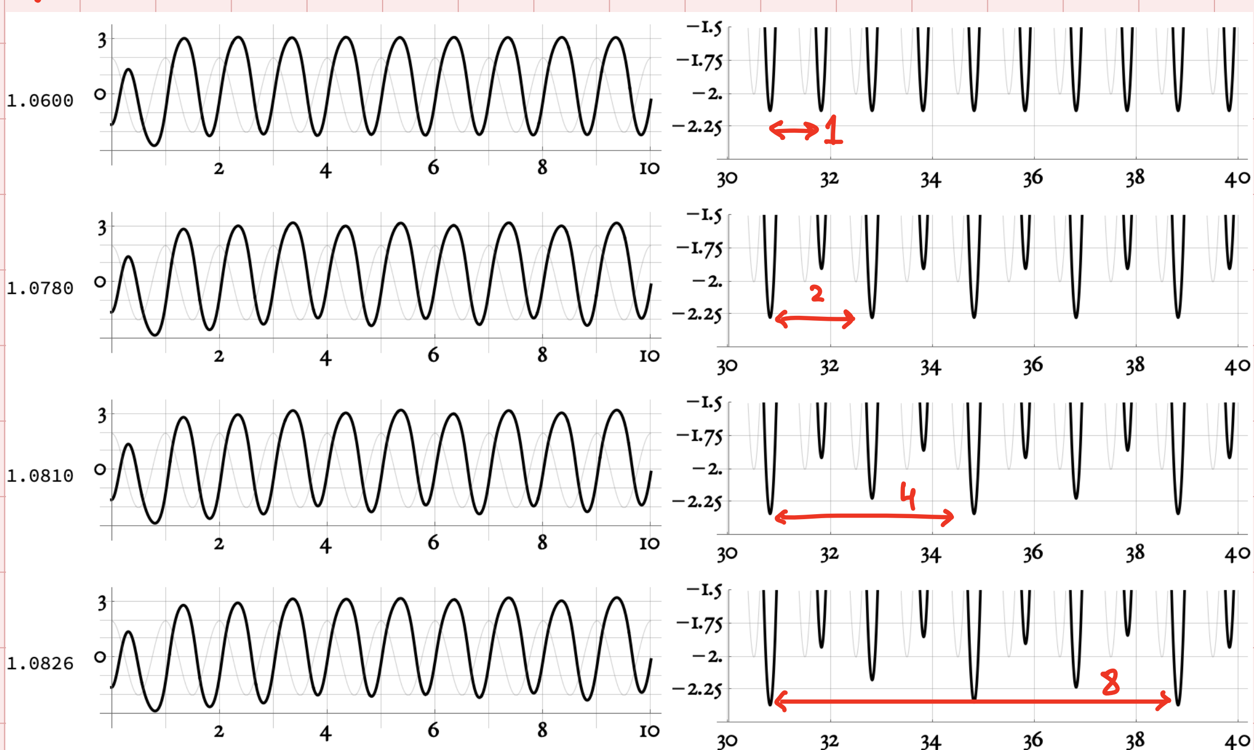

Damped Driven Pendulum γ = 0.9
Close–up compared to pure sinusoid


Damped Driven Pendulum γ = 1.06
Close–up of later time

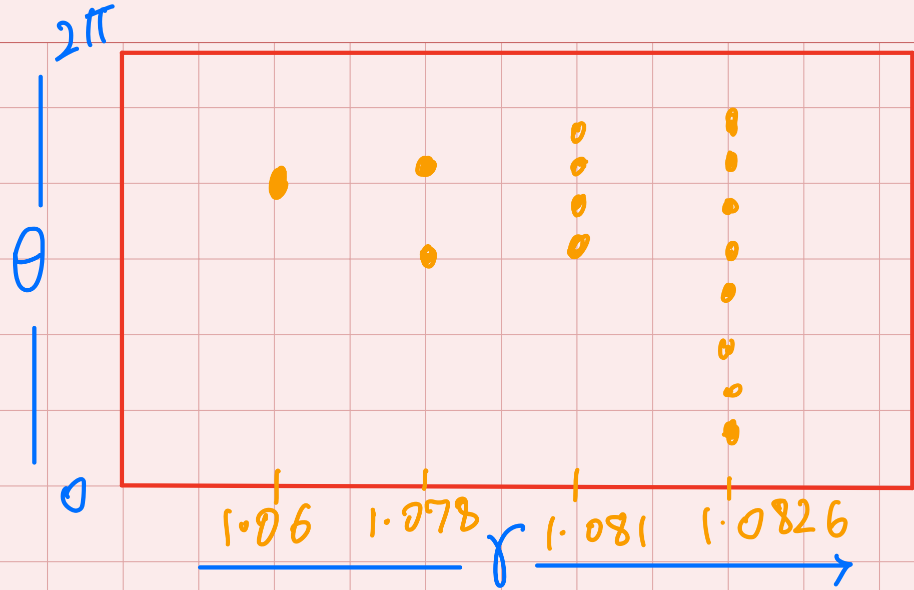

Damped Driven Pendulum γ = 1.073
Close–up of periodic behavior



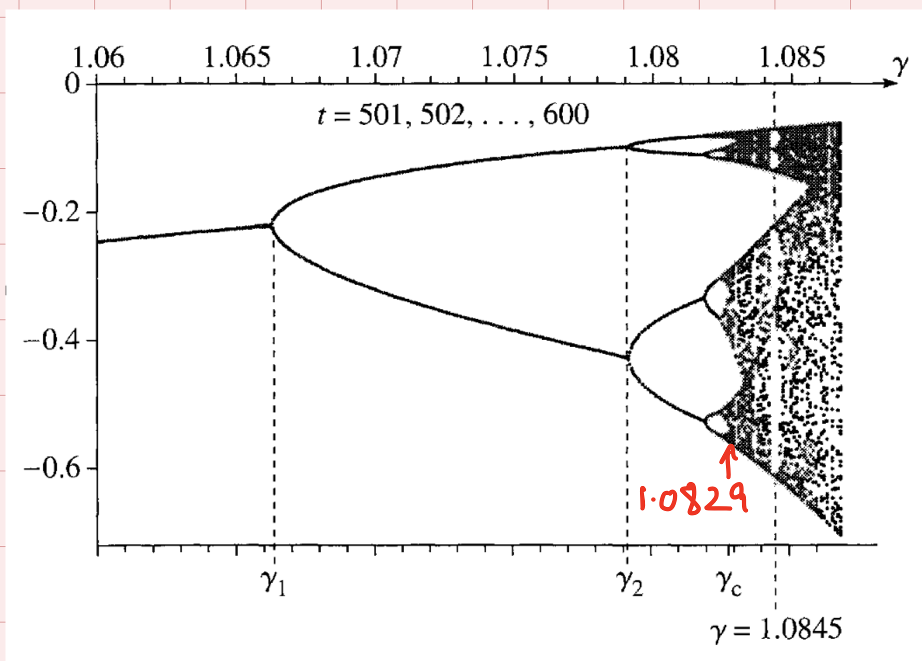
γ θ vs. time



Orbit diagram: For each γ, start with same initial condition. Integrate to find $\theta(t)$ to long times. Check the value of θ every 1 time unit. Organize results into an "orbit diagram". e.g. Plot the last 1000 time units.



$\gamma = 1.06$



Period-doubling route to chaos.

$\gamma_1 \longrightarrow \gamma_2 \longrightarrow \gamma_3 \longrightarrow \cdots$

$$\lim_{n\to\infty} \frac{\gamma_{n+1} - \gamma_n}{\gamma_n - \gamma_{n-1}} = \frac{1}{\delta} \quad , \quad \boxed{\delta = 4.669}$$

Wed, Apr 30 Lecture 26

→ Nonlinear differential equations describe most of the physical world. Linear equations are usually approximations e.g. spring close to rest length; small deflections of a beam; constant 'g' for gravity.

→ The number of dimensions of phase space needed to describe real systems' dynamics can get very large. (degrees of freedom) Explicit time dependence adds another dimension to phase space. Most differential equations in physics are 2nd or 4th order.

→ Nonlinearity + "high"-dimensionality ⟶ possibility of chaos. How nonlinear? e.g. A pendulum to large angles $\theta \to \sin\theta$
How high-dimensional? $n > 2$ is enough.

→ Partial differential equations are like ∞-dimensional ordinary differential equations.

→ In the 19th century, it was thought that physics, in principle, had been solved. Initial conditions and differential eqns in ⟶ future behavior out.
"Laplace's Demon"

→ Quantum physics and general relativity complicated this picture at v. small and v. large scales. But until the discoveries of Lorenz in the 1960s, people thought that classical physics, on which engineering is based, did not suffer from such 'problems'.

→ The discovery of chaos — strictly a mathematical phenomenon — in numerous physically important equations has broken the spell of Laplace's Demon. It turns out that an adequate knowledge of the initial conditions is not enough to guarantee adequate knowledge of future behavior. — even if you know the governing equations and their solutions are guaranteed to exist and be unique.

→ The degree of unpredictability is so great that chaotic systems — despite being deterministic, not random — show certain features of random processes. e.g. long-time behavior of logistic map resembles long random walks.

→ We therefore learn that there is often a pretty stark limit on how long we can predict physical phenomena for, <u>even if the physics is well-understood.</u> Small uncertainties in measurement will propagate exponentially. Better measurement tools only delay this inevitable divergence of initially nearby trajectories very slightly.

→ So, the question of whether we live in a deterministic world is complicated by the presence of chaos in those equations that we know to be good models of physical phenomena. (Sometimes, even in the simplest eqs.) Even if the world is deterministic — i.e. even if the future state of the world is a function of current state of the world — it often appears to behave randomly if chaos is present in the governing equations.