A New Definition of Vortex Formation Length

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Abstract We introduce a new definition of vortex formation length for uniform steady flow past a circular cylinder using Proper Orthogonal Decomposition (POD) of the pressure field. Previous definitions each identify a single characteristic length for a given value of Reynolds number. We use the leading modes of the pressure POD to define upper and lower bounds on the vortex formation length. Identifying a range for the vortex formation length is consistent with the hysteresis observed in the critical spacing of two tandem cylinders.

Keywords Bluff body wakes · Vortex dynamics · Vortex formation length

1 Introduction

A key feature of bluff body wakes is the periodic formation of shed vortices. As suggested by Roshko [1] and illustrated in Fig. 1a, for a single fixed cylinder with diameter D in steady, uniform flow having free-stream speed U_∞ and kinematic viscosity ν one can use the time-averaged flow field to identify the non-dimensional *vortex formation length*, $\ell_v = L_v/D$, at which individual vortices have completed their formation and are released into the wake. The established definitions of ℓ_v [2] succeed in capturing time-averaged trends, such as the fact that vortices are released closer to the bluff body as Reynolds number, $Re = U_\infty D/\nu$, is increased. However, these definitions do not make use of important temporal information in the flow field. Here we consider a new approach; for brevity and simplicity, we will focus

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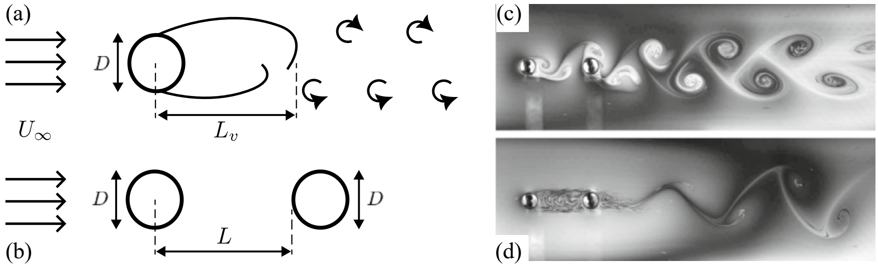

Fig. 1 **a** Schematic of the vortex formation length, $L_v = \ell_v D$, for a single fixed cylinder with diameter D. **b** The spacing, $L = \ell D$, of two tandem cylinders as defined here for comparison with L_v. **c**, **d** Examples of **c** the 'wake-in-the-gap' structure and **d** the 'extended body' structure in the flow past two tandem cylinders, reproduced from [4]; flow is from left to right.

our attention on $Re = 100$, a commonly considered case exhibiting predominantly two-dimensional flow [3].

The flow quantity used to determine ℓ_v varies across the literature. For $Re = 100$, *maximum pressure* has given $\ell_v \approx 1.4$ [1], *maximum fluctuating kinetic energy* has given $\ell_v \approx 2.8$ when looking only along the centerline and $\ell_v \approx 3.0$ when considering the entire field [2], and *maximum fluctuating streamwise velocity* has given $\ell_v \approx 3.0$ when looking only along the centerline and $\ell_v \approx 2.2$ when considering the entire field [2]. We will focus here on just the fluctuating pressure field. A study comparing these results with those for fluctuating streamwise velocity and kinetic energy across a range of Reynolds numbers will be the subject of a future manuscript.

When two identical cylinders are aligned in tandem, as in Fig. 1b, the dimensionless spacing, $\ell = L/D$, affects the flow structure and the fluid forces on the cylinders [5]. If ℓ is large, the wake of the upstream cylinder resembles the classic '2S' wake [3], as in Fig. 1c. For close spacing, as in Fig. 1d, there occurs an "extended body" flow structure. There is a *critical spacing*, ℓ_c, of two inline cylinders at which the flow transitions between the 'wake-in-the-gap' structure and the 'extended body' structure.[1] The value of ℓ_c exhibits hysteresis when the system is varied quasi-statically by either increasing or decreasing the Reynolds number or the cylinder spacing [4]. At $Re = 100$, for example, it has been observed that the bounds on the critical spacing are $\ell_{c,min} \approx 2.5$ and $\ell_{c,max} \approx 3.5$ [4]. The hysteresis observed in the critical spacing of tandem cylinders suggests that there exists a range of spatial locations behind a single cylinder that is crucial to the development of the wake's vortical structure.

We propose a new definition of vortex formation length that produces a spatial range, not a single value, at a given Reynolds number. The methods used in our analysis are presented in Sect. 2, and in Sect. 3 we focus our vortex formation length analysis on $Re = 100$ using 2D computational simulations. We conclude in Sect. 4.

[1] The standard definition of ℓ_c uses center-to-center distance, giving a value that is 0.5 larger than what is used here. The current definition allows for direct comparison with ℓ_v.

2 Methods

Results were obtained by solving the 2D Navier–Stokes equation with the Lattice-Boltzmann method using the BGK scheme [6] with the D2Q9 model and the single relaxation time approximation [7]. The $100D \times 100D$ rectangular domain, with the cylinder(s) positioned in the center, was meshed using a multi-block grid refinement method [8]. The upstream boundary condition was uniform flow with speed U_∞. The outlet boundary condition on the vector velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ was $\partial \boldsymbol{u}/\partial t + U_\infty\, \partial \boldsymbol{u}/\partial x = 0$. The Neumann boundary condition $\partial \boldsymbol{u}/\partial n = 0$ was used on the sides of the domain, with n the outward-normal coordinate. On the cylinder boundary we specified $\boldsymbol{u}(\boldsymbol{x}, t) = 0$ using Bouzidi's [9] method. Differences between our results and several previous works (e.g., [6]) for average drag coefficient, average lift coefficient, and Strouhal number for uniform flow past a single cylinder are within 5%.

Our analysis is based on Proper Orthogonal Decomposition (POD) [10] of the fluctuating scalar pressure field $p'(\boldsymbol{x}, t) = p(\boldsymbol{x}, t) - \overline{p}(\boldsymbol{x})$, where $\overline{p}(\boldsymbol{x})$ is the time-averaged pressure field. We applied this decomposition to $p'(\boldsymbol{x}, t)$ in the spatial window bounded by $0.5D \le x \le 12D$ and $-2D \le y \le 2D$, where $(x, y) = (0, 0)$ is at the cylinder's center. We generated each data matrix $\boldsymbol{Q} = \left[p'(\boldsymbol{x}, t_1)\ p'(\boldsymbol{x}, t_2) \ldots p'(\boldsymbol{x}, t_N)\right]$ using $N = 1000$ time snapshots spanning 10 periods of vortex shedding. The POD modes $\phi_n(\boldsymbol{x})$ are given by the eigenbasis of the matrix $\boldsymbol{K} = \boldsymbol{Q}\boldsymbol{Q}^T$, with modes ordered by level of decreasing contribution to the flow energy. The time-invariant modes $\phi_n(\boldsymbol{x})$ contain important information about the underlying structure of the time-varying pressure field, which is reconstructed from the modes by $p'(\boldsymbol{x}, t) = \sum_{n=1}^{N} a_n(t)\, \phi_n(\boldsymbol{x})$.

3 A New Definition of Vortex Formation Length

We present a new approach to defining the vortex formation length, based here on a POD analysis of the fluctuating pressure field. A similar approach can be applied to the fluctuating kinetic energy or streamwise velocity fields.

The formation and release of a vortex is a strongly time-dependent process. Reconstructing the flow field using only the first few POD modes can 'filter out' the effects of small-scale unsteadiness and help reveal important spatial structures. The formation of a vortex every half-period is one of the most significant and 'high-energy' events that occurs in the near-wake region, and thus the dominant POD modes should provide key information regarding the position(s) at which a vortex is formed in the wake.

Figure 2e shows the energy associated with the first 10 modes in the pressure-based POD representation of the flow past a circular cylinder at $Re = 100$. The first two modes contain 86% of the total energy, and modes 3 and 4 contain another 13% of the energy. The spatial structure of each of the first four modes is shown in Fig. 2a–d.

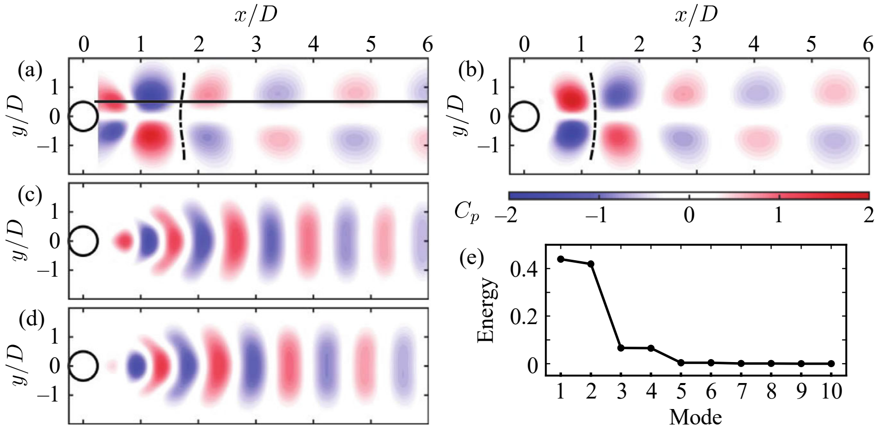

Fig. 2 **a** First, **b** second, **c** third, and **d** fourth POD modes of $p'(\boldsymbol{x}, t)$. Solid horizontal line in **a** is the line at $y/D = 0.5$ used to generate C_p curves in Fig. 3. Dashed lines in **a**, **b** show $\ell_{0,i}$. **e** Energy for the first 10 modes of the fluctuating pressure field

We determine a range of vortex formation length as follows. First, we select a representative line parallel to the wake centerline that intersects the relevant wake structure; the example $y/D = 0.5$ is shown in Fig. 2c. We then extract the non-dimensional fluctuating pressure along this line, giving $C_p = 2(p - p_\infty)/\rho U_\infty^2$ as a function of x/D for each of the modes, as in Fig. 3. Modes 1 and 2 show a global maximum in C_p for $x/D < 3$, with C_p settling into a decaying, nearly sinusoidal pattern for $x/D \gtrsim 5$. For modes 3 and 4, this peak occurs near $x/D \approx 5$. The end of vortex formation in a given mode is taken to be the first zero of C_p after the global maximum; we will refer to this (dimensionless) position as $\ell_{0,i}$, where i identifies the POD mode. Conceptually, $\ell_{0,i}$ corresponds to the streamwise position beyond which mode i exhibits a decaying wake pattern, and we suggest that $\ell_{0,i}$ is the spatial location at which mode i stops contributing significantly to vortex formation.

Since in this example most of the flow energy is contained within the first two POD modes, the structure contained within these two modes dominates the wake dynamics. We define the lower bound on the vortex formation length, $\ell_{v,min}$, to be

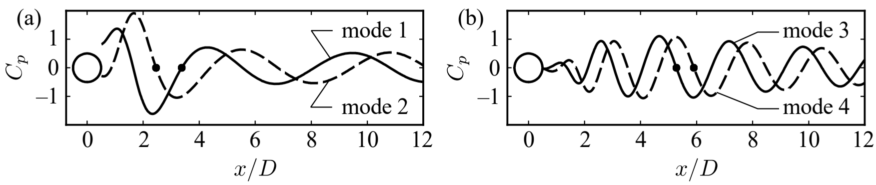

Fig. 3 Pressure coefficient along the line $y/D = 0.5$ as a function of streamwise distance from the center of the cylinder for **a** the first two modes and **b** the third and fourth modes of the pressure field. The solid circles indicate the locations of $\ell_{0,i}$ for each of the modes

the smaller value of $\ell_{0,i}$ (for $i = 1, 2$), which here occurs for mode 2, and we take the upper bound $\ell_{v,max}$ be the larger value of $\ell_{0,i}$, here from mode 1. The dashed lines in Fig. 2a, b show the location of $\ell_{0,i}$ as a function of y/D; these values are only weakly dependent on y/D. Thus, for $Re = 100$ we define the vortex formation length behind a circular cylinder to be bounded by the values $\ell_{v,min} = \ell_{0,2} \approx 2.5$ and $\ell_{v,max} = \ell_{0,1} \approx 3.4$.

The POD modes 3 and 4 indicate that approximately 13% of the flow energy contributes to a wake structure that continues strengthening, in the sense of increasing peaks in the fluctuating pressure field, until $x/D \approx 5.5$. The location at which these two low-energy modes show a transition from formation to dissipation is well beyond the expected value for the time-averaged vortex formation length [2] according to any of the existing definitions. This observation supports our assumption that here the two leading modes are the key modes for characterizing the vortex formation length.

4 Conclusions

Important information about the spatial extent of vortex formation behind a bluff body in cross flow is gained from considering only the time-averaged flow, but significant information is lost in the standard calculation of ℓ_v by neglecting the fluctuating field quantities. We have used Proper Orthogonal Decomposition (POD) to identify a *range* of vortex formation length based on critical points in the fluctuating pressure field. At $Re = 100$, we find $2.5 \lesssim \ell_v \lesssim 3.4$, a range of values that is consistent with many of the known time-averaged results for ℓ_v [2].

We stated in Sect. 1 that the hysteresis observed in the critical spacing between two tandem cylinders motivated considering a new definition for ℓ_v. At $Re = 100$, observed bounds on the critical spacing are $\ell_{c,min} \approx 2.5$ and $\ell_{c,max} \approx 3.5$ [4]. A direct comparison of the bounds on ℓ_c with those we observed for ℓ_v suggests that our definition does provide important new information. In the case of two tandem cylinders, positioning the downstream cylinder with spacing $\ell_{v,min} \lesssim \ell \lesssim \ell_{v,max}$ would cause interference with one of the two primary POD modes involved in the vortex formation process, while a spacing $\ell \lesssim \ell_{v,min}$ interferes with both modes. This perspective suggests that just one unimpeded mode is necessary to sustain an existing wake-in-the-gap flow, but both high-energy modes are necessary for initiating that flow structure.

Our analysis here is limited in scope: it is specific to a 2D simulation of a circular cylinder at $Re = 100$, we have considered only a POD analysis of the flow, and we have based our definition on just the pressure field. Relevant applications occur at a wide range of Reynolds numbers, there are other flow decompositions that may be used, and most prior calculations of vortex formation length use the fluctuating velocity field [2]. The extension of our proposed approach in these directions is an open question. Given the fundamental nature of vortex wake formation, we anticipate that this overall approach—of using a flow decomposition to identify from a fluctuating field quantity the critical bounds on vortex formation length—will be relevant to a wide variety of applications and open to numerous implementation approaches.

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