Sparse optimization Algorithms for Dynamic Imaging

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Sparse optimization Algorithms for Dynamic Imaging

based on joint works with

Kristian Bredies, Marcello Carioni, Francisco Romero, Daniel Walter

Outline

1. Minimization Problem / Sparsity
2. Algorithm for sparse solutions
3. Dynamic Imaging: Particle Tracking problem

Minimization Problem

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X Banach space. Solve

$$\min_{u \in X} \; L(u) + R(u)$$

- **L** $\rightsquigarrow$ **Loss function:** **Smooth** $+$ **Convex**

  $$L : X \to [0, \infty)$$ (Close to data)

- **R** $\rightsquigarrow$ **Regularizer:** **Convex** $+$ **1-homogeneous**

  $$R : X \to [0, \infty]$$ (Promotes Sparsity)

Note: Compactness assumptions $\implies$ Minimizer exists

[1] Bredies, Carioni, Fanzon, Walter. **Mathematical Programming** (2023)

Sparsity

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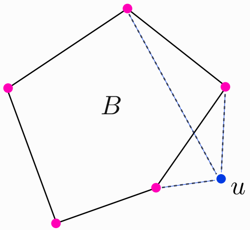
Unit Ball of regularizer R

$$B := \{u \in X : \; R(u) \leq 1\}$$

Extremal Points: $u \in B$ s.t.

$$\begin{cases} u = \alpha u_1 + (1-\alpha) u_2 \\ \alpha \in (0,1)\,, \; u_1, u_2 \in B \end{cases} \implies \quad u = u_1 = u_2$$



Definition: $u \in X$ **sparse**

Conic combination

$$u = \sum_{i=1}^{N} \lambda_i \, u_i \,, \quad \lambda_i \geq 0 \,, \quad u_i \in \mathrm{Ext}(B)$$

[1] Bredies, Carioni, Fanzon, Walter. **Mathematical Programming** (2023)

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Numerical **Algorithm** to compute

$$\bar{u} \in \arg\min_{u \in X} \; L(u) + R(u)$$

which is **sparse**

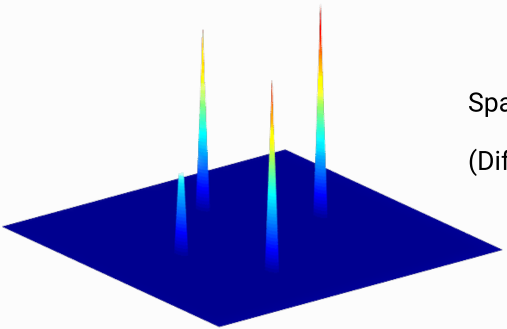
$$\bar{u} = \sum_{i=1}^{N} \lambda_i \, u_i \,, \quad \lambda_i \geq 0 \,, \quad u_i \in \mathrm{Ext}(B)$$

Examples:

- Portfolio optimization $\rightsquigarrow$ $\mathbb{R}^d$
- Training 2-Layer Neural Networks $\rightsquigarrow$ $\mathcal{M}(\mathbb{R}^d)$ Radon Measures
- Microstructures in Materials $\rightsquigarrow$ $\mathrm{BV}(\mathbb{R}^d)$ Bounded Variation

Example: Radon Measures

Banach space: $X = \mathcal{M}(\mathbb{R}^d)$ Radon measures



Sparse Source Identification

(Diffusion-Advection Equation)

$$\mu = \sum_i \lambda_i \, \delta_{x_i}$$

Regularizer: $R(\mu) := \|\mu\|_{\mathcal{M}}$ $\mathrm{Ext}(B) = \{\pm\delta_x \, : \, x \in \mathbb{R}^d\}$

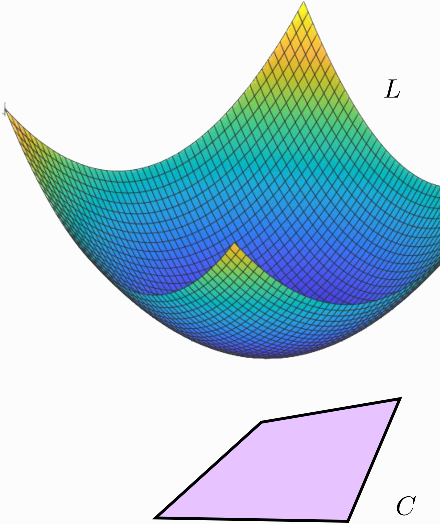
Figure from: Monge, Zuazua. **Systems** & **Control Letters** (2020)

Algorithm: Classic Frank-Wolfe

Problem: Constrained minimization

$$\min_{x \in C} L(x)$$

- $L : \mathbb{R}^N \to \mathbb{R}$ regular convex
- $C \subset \mathbb{R}^N$ convex compact set



M. Jaggi. **Proceedings of Machine Learning Research** (2013)

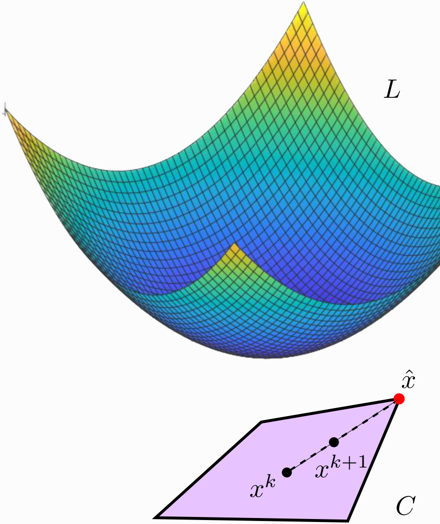
Frank-Wolfe Algorithm: Given $x^k \in C$

1. **Insertion:** Solve linearized problem

$$\min_{x \in C} \langle \nabla F(x^k), x \rangle \quad \mapsto \quad \widehat{x}$$

2. **Convex update**: Set

$$x^{k+1} := x^k + \alpha(\widehat{x} - x^k)\,, \quad \alpha := \frac{2}{k+2}$$



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Algorithm: Generalized Frank-Wolfe

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$$\min_{u \in X} G(u)\,, \quad G := L + R$$

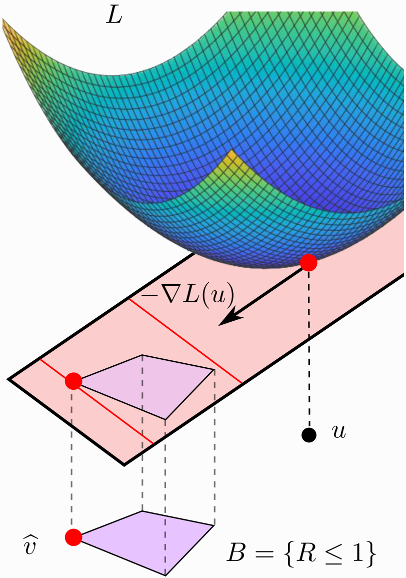
Idea: Set $B = \{R \le 1\}$. Consider

$$\min_{u \in X} L(u) + \chi_B(u) \iff \min_{u \in B} L(u)$$

Descent Direction: Solve

$$\min_{v \in B} \langle \nabla L(u), v \rangle \quad \mapsto \quad \hat{v}$$

Lemma [1]. $\hat{v} \in \operatorname{Ext}(B)$



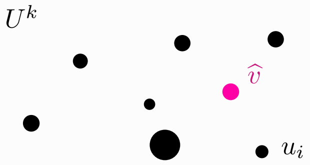
[1] Bredies, Carioni, **Fanzon**, Walter. **Mathematical Programming** (2023)

Algorithm: Generalized Frank-Wolfe

Sparse k-th iterate

$$U^k = \sum_{i=1}^{n} \lambda_i \, u_i$$

$$\lambda_i \geq 0, \quad u_i \in \mathrm{Ext}(B)$$



❶ **Insertion Step:** Solve

$$\widehat{v} \in \operatorname*{arg\,min}_{v \in \mathrm{Ext}(B)} \langle \nabla L(U^k), v \rangle$$

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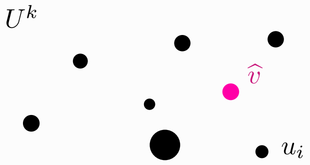

1. **Insertion Step:** Solve

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2. **Fully-Corrective Step:** Set $u_{n+1} := \widehat{v}$

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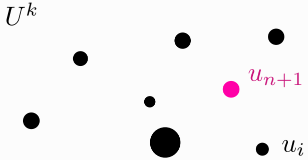

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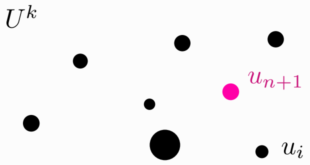

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$$(\lambda_1^*, \ldots, \lambda_{n+1}^*) \in \operatorname*{arg\,min}_{\lambda_i \geq 0} G\left(\sum_{i=1}^{n+1} \lambda_i \, u_i \right) \quad \rightsquigarrow \quad U^{k+1} := \sum_{i=1}^{n+1} \lambda_i^* \, u_i$$

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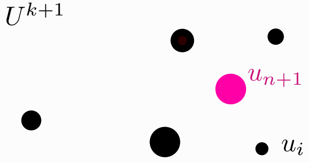

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Theorem [1]

U^k sparse iterate generated by Algorithm. Then

$$U^k \stackrel{*}{\rightharpoonup} \bar{u} \,, \qquad \bar{u} \in \arg\min G \,, \quad G := L + R$$

General convergence is **sublinear**

$$G(U^k) - \min G \lesssim \frac{1}{k}$$

Highlight: $\bar{u}$ **sparse** $+$ **"source condition"** $\Longrightarrow$ **linear** convergence

$$G(U^k) - \min G \lesssim \frac{1}{2^k}$$

[1] Bredies, Carioni, **Fanzon**, Walter. **Mathematical Programming** (2023)

Linear Convergence Assumptions

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1. Minimizer of $G := L + R$ is **sparse**

$$\bar{u} = \sum_{i=1}^{M} \bar{\lambda}_i \, \bar{u}_i \, , \qquad \bar{u}_i \in \mathrm{Ext}(B)$$

2. **Source condition:** dual variable

$$\bar{p} := -\nabla L(\bar{u})$$

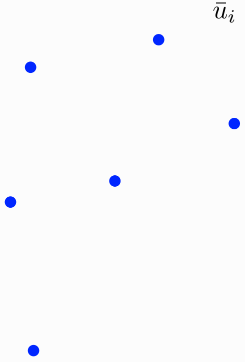
is maximized exactly at $\bar{u}_i$

$$\operatorname*{arg\,max}_{v \in \mathrm{Ext}(B)} \langle \bar{p}, v \rangle = \{\bar{u}_1, \dots, \bar{u}_M\}$$

3. **Quadratic growth** of $\bar{p}$ around $\bar{u}_i$

$$1 - \langle \bar{p}, u \rangle \gtrsim g(u, u_i)^2 \, , \quad u \sim u_i$$

where g is "distance" on $\mathrm{Ext}(B)$



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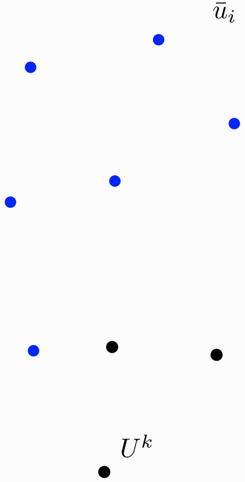

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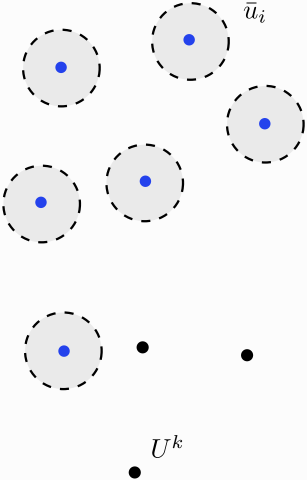

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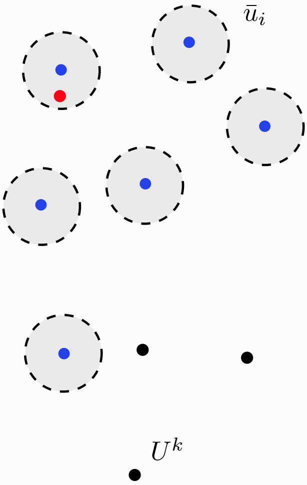

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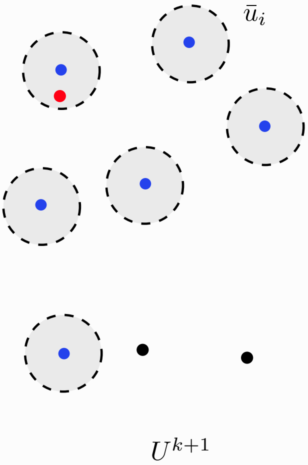

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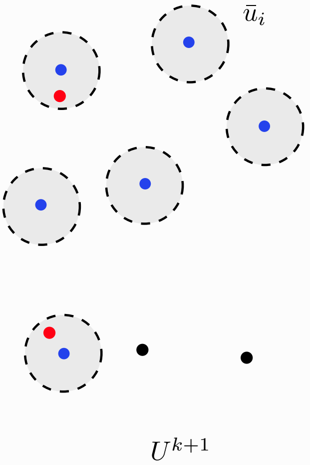

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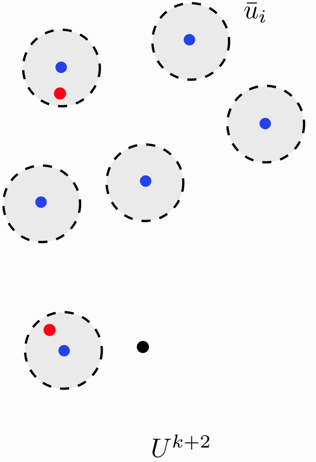

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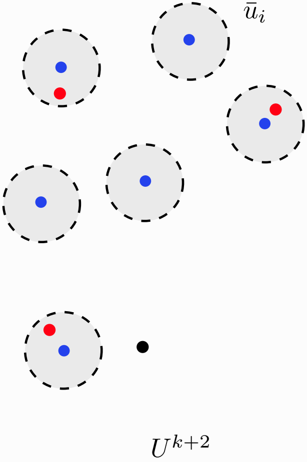

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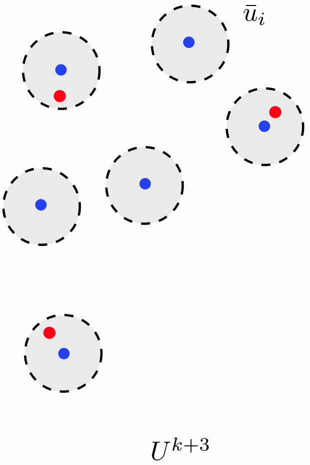

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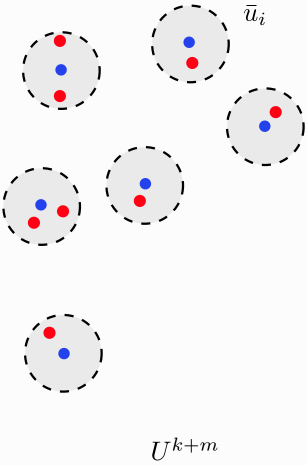

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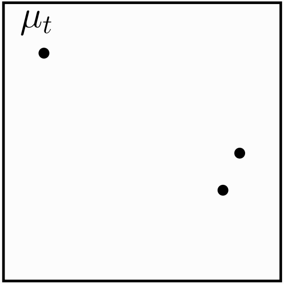
[1] Bredies, Carioni, Fanzon, Walter. **Mathematical Programming** (2023)

Application: Dynamic MRI

y_t

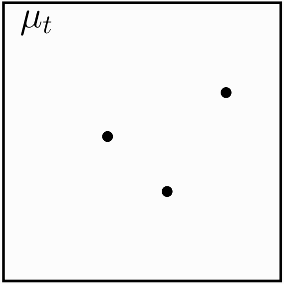


$$\mu_t = \sum_{i=1}^{3} \delta_{x_i(t)}$$

Frame-by-Frame: MRI Data $y_t \rightsquigarrow$ Image μ_t

Application: Dynamic MRI

y_t



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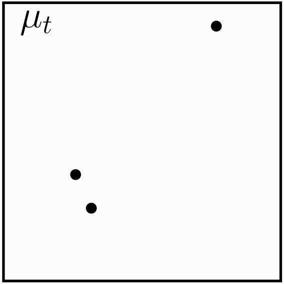

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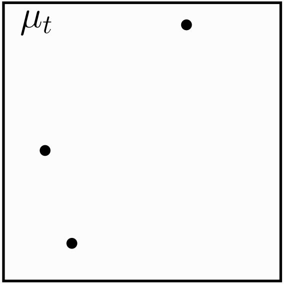

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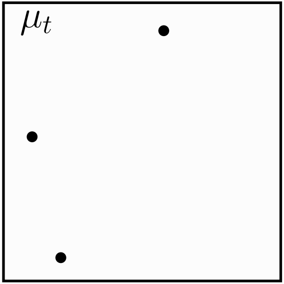

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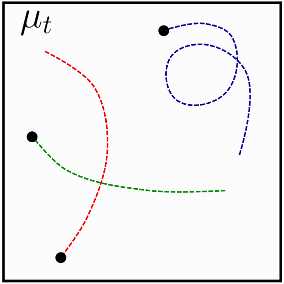
y_t



$$\mu_t = \sum_{i=1}^{3} \delta_{x_i(t)}$$

Frame-by-Frame: MRI Data $y_t \rightsquigarrow$ Image $\mu_t \implies$ Interpolate Trajectories

Application: Dynamic MRI

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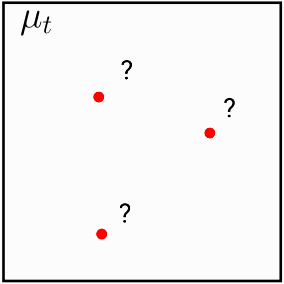
y_t



$$\mu_t = \sum_{i=1}^{3} \delta_{x_i(t)}$$

Frame-by-Frame: MRI Data $y_t \rightsquigarrow$ Image $\mu_t \implies$ Interpolate Trajectories

Issue: Motion $\implies$ Low Scan Time $\implies$ **Low Data** $\mathbf{y}_t \rightsquigarrow$ **Particles?**

Global-in-Time: Full Time-Series $t \mapsto y_t \rightsquigarrow$ Trajectories $t \mapsto \mu_t$

Framework:

- **Image frame:** Spatial domain $\Omega \subset \mathbb{R}^N$
- **Images:** Modelled as Radon Measures $\mu \in \mathcal{M}(\Omega)$
- **Data spaces:** Hilbert spaces H_t for $t \in [0,1]$
- **Measurement Operators:** linear continuous maps

$$K_t \colon \mathcal{M}(\Omega) \to H_t$$

- **Data points:** Curve $t \mapsto y_t$ with $y_t \in H_t$

Dynamic Inverse Problem: Find **curve** of measures $t \mapsto \mu_t \in \mathcal{M}(\Omega)$ s.t.

$$K_t \mu_t = y_t \quad \text{for all} \quad t \in [0,1]$$

[2] Bredies, **Fanzon**. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)

Motion-Aware Regularization

Trajectories: Curve of measures

$$t \mapsto \mu_t \in \mathcal{M}(\Omega)\,, \qquad t \in [0,1]$$

Assumptions:

- μ_t satisfies **Continuity Equation**

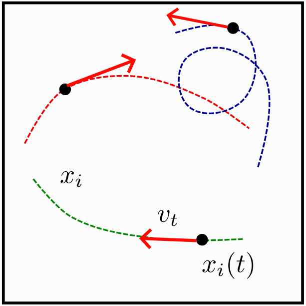
  $$\partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0$$

  for some velocity field (to find)

  $$v_t : \mathbb{R}^2 \to \mathbb{R}^2$$

- Finite **Kinetic Energy**

  $$\int_0^1 \int_{\mathbb{R}^2} |v_t(x)|^2 \, d\mu_t(x) \, dt < \infty$$



$$\mu_t = \sum_i \delta_{x_i(t)}$$

[2] Bredies, Fanzon. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)

Motion-Aware Regularization

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Minimization Problem: Given data $t \mapsto y_t \in H_t$

$$K_t \mu_t = y_t \quad \rightsquigarrow \quad \min_{\mu, v} \; L(\mu) + R(\mu, v)$$

- $L \rightsquigarrow$ **Loss Function:** Fits $t \mapsto \mu_t$ to given data $t \mapsto y_t$

$$L(\mu) := \int_0^1 \|K_t \mu_t - y_t\|_{H_t}^2 \; dt$$

- $R \rightsquigarrow$ **Regularizer:**

$$R(\mu, v) := \underbrace{\int_0^1 \int_{\mathbb{R}^2} |v_t(x)|^2 \; d\mu_t(x) \, dt}_{\text{Kinetic Energy}} \quad \text{s.t.} \quad \underbrace{\partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0}_{\text{Continuity Equation}}$$

Note: R is connected to **Optimal Transport** (Benamou-Brenier formula)

[2] Bredies, **Fanzon**. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)

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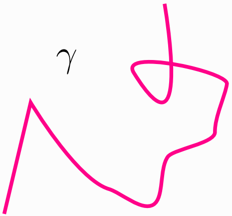
$$R(\mu, v) := \int_0^1 \int_{\mathbb{R}^2} |v_t(x)|^2 \, d\mu_t(x)\, dt \quad \text{s.t.} \quad \partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0$$

Theorem [3]

Let $B = \{R \le 1\}$. Then $\operatorname{Ext}(B)$ are measures $t \mapsto \mu_t$ supported on **Sobolev Curves**

$$t \mapsto \mu_t = \delta_{\gamma(t)}\,, \qquad \gamma \in H^1([0,1];\mathbb{R}^2)$$



Proof Idea: Probabilistic Superposition Principle for measure solutions to

$$\partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0 \qquad (= g_t \mu_t)$$

[3] Bredies, Carioni, **Fanzon**, Romero. **Bulletin London Mathematical Society** (2021)
[4] Bredies, Carioni, **Fanzon**. **Communications in PDEs** (2022)

Application: Dynamic Undersampled MRI

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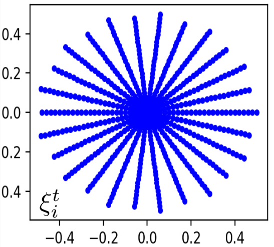
Fourier Transform: For $\mu \in \mathcal{M}(\Omega)$

$$\hat{\mu}\colon \mathbb{C} \to \mathbb{C}\,, \quad \hat{\mu}\,[\xi] := \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i\xi\cdot x}\, d\mu(x)$$

Sampling Frequencies: M_t time dependent points

$$\xi_1^t\,, \dots, \xi_{M_t}^t \in \mathbb{C}$$



Radial Sampling

Forward operators: linear continuous

$$K_t\colon \mathcal{M}(\Omega) \to \mathbb{C}^{M_t}\,, \quad K_t\mu := \left(\hat{\mu}[\xi_1^t]\,, \dots, \hat{\mu}[\xi_{M_t}^t]\right)$$

Dynamic MRI IP: Given $t \mapsto \mu_t \in \mathbb{C}^{M_t}$ find $t \mapsto \mu_t \in \mathcal{M}(\Omega)$ s.t.

$$K_t\mu_t = y_t \quad \text{for all} \quad t \in [0,1]$$

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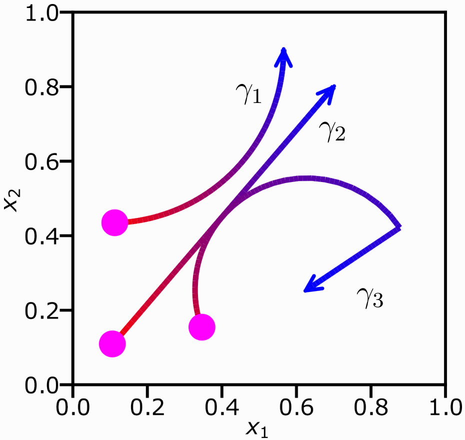
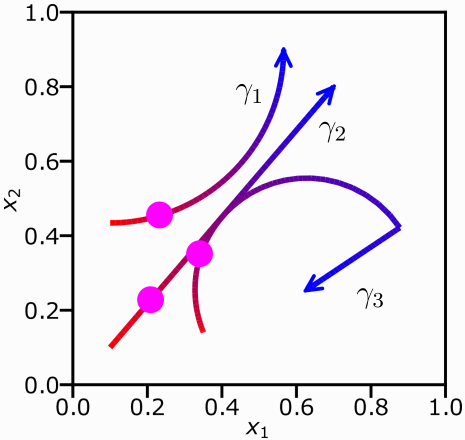


Ground truth: Curve of measures

$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$



Ground truth: Curve of measures

$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

Dynamic MRI: Experiment

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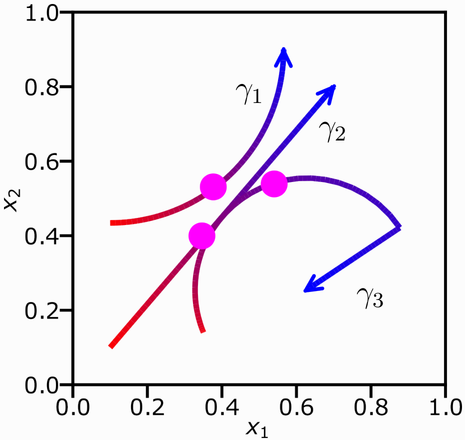

Ground truth: Curve of measures

$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

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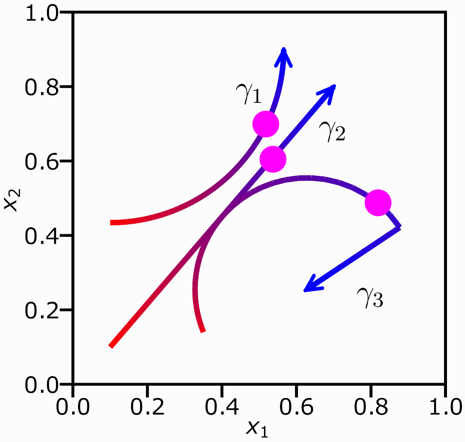

Ground truth: Curve of measures

$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

Dynamic MRI: Experiment

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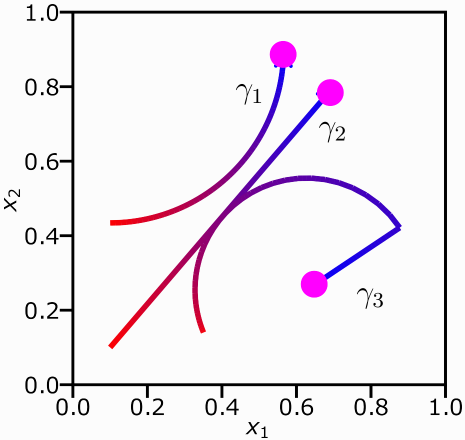

Ground truth: Curve of measures

$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

Dynamic MRI: Experiment

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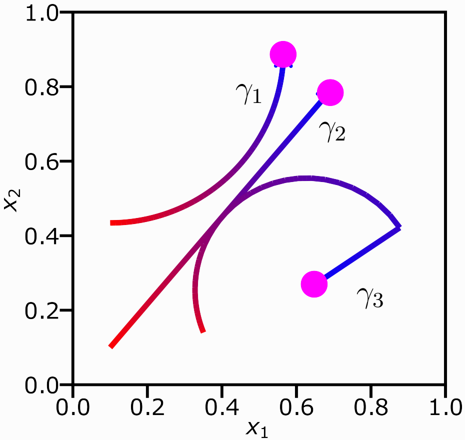

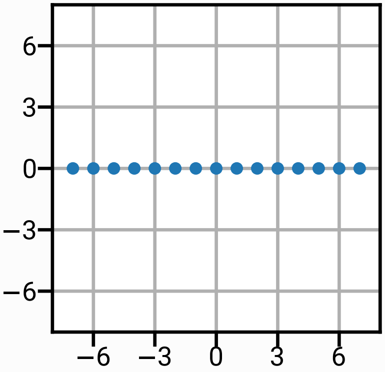

$t = 0$

Ground truth: Curve of measures

$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

Dynamic MRI: Experiment

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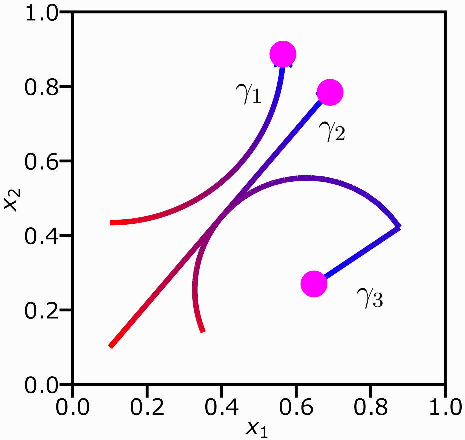
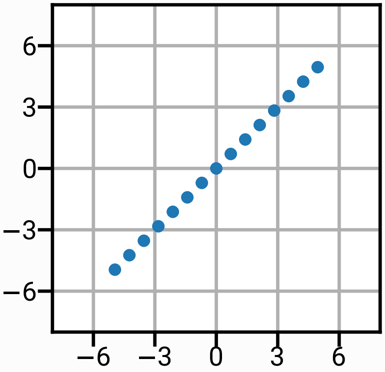
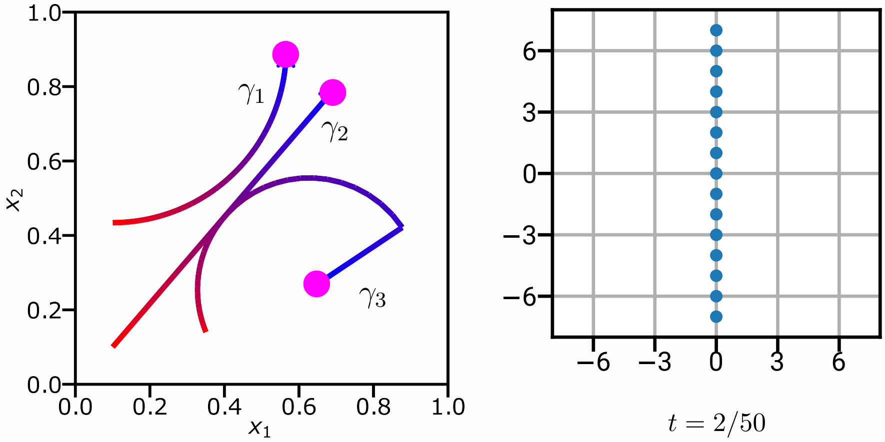




$t = 1/50$

Ground truth: Curve of measures

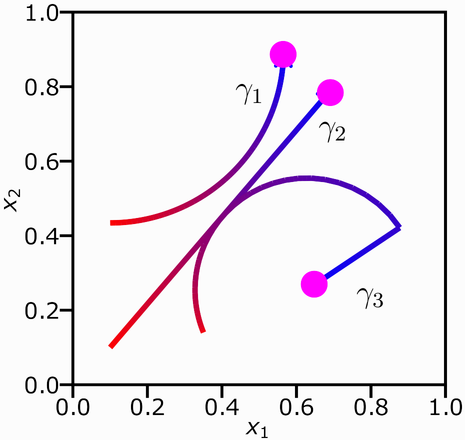
$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

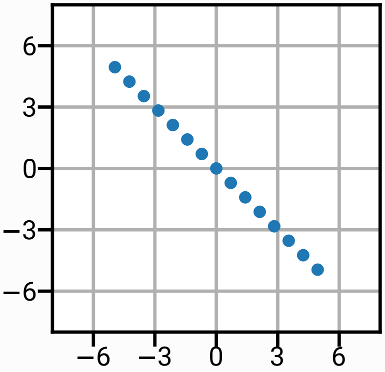


$t = 2/50$

Ground truth: Curve of measures

$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

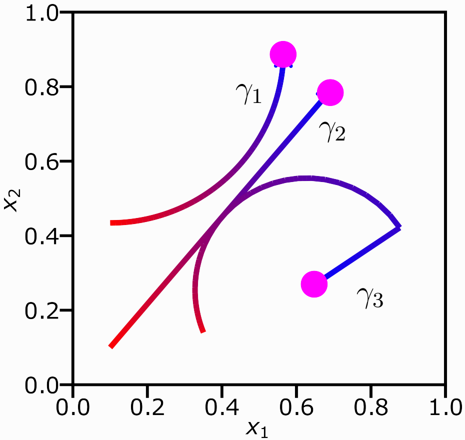


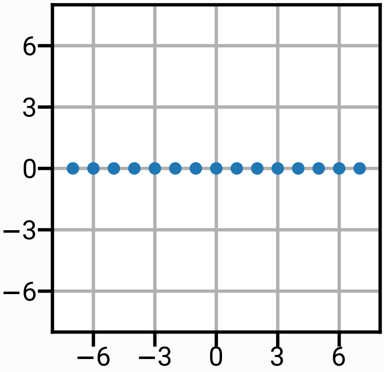


$t = 3/50$

Ground truth: Curve of measures

$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$





$t = 4/50$

Ground truth: Curve of measures

$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

Dynamic MRI: Experiment

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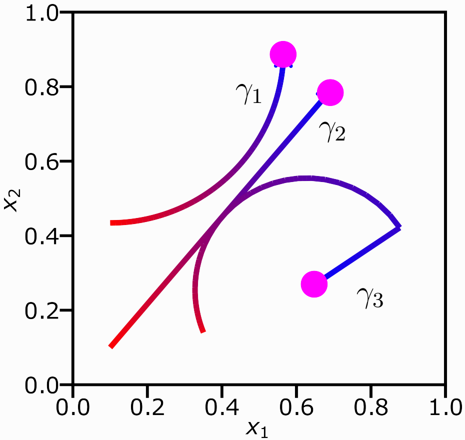
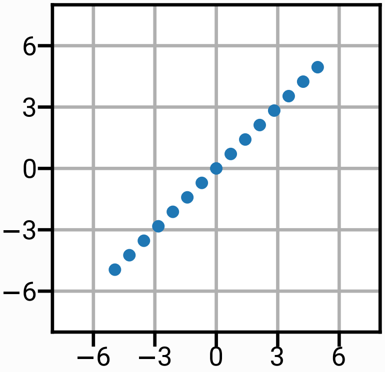


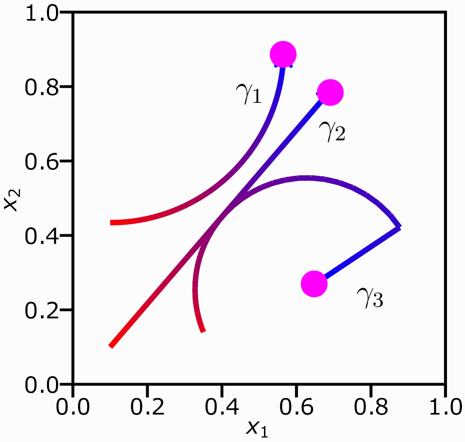


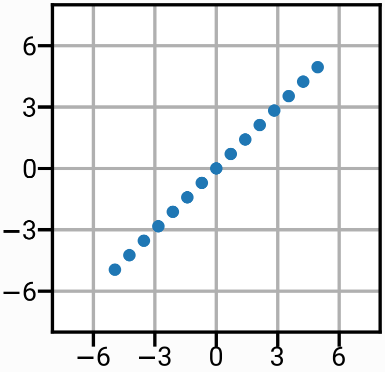
$$t = 5/50$$

Ground truth: Curve of measures

$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

Dynamic MRI: Experiment





$t = 5/50$

Ground truth: Curve of measures

$$\bar{\mu}_t := \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$

Data: Defined by

$$y_t := K_t \mu_t + 20\% \text{ Noise}$$

Dynamic MRI: Experiment

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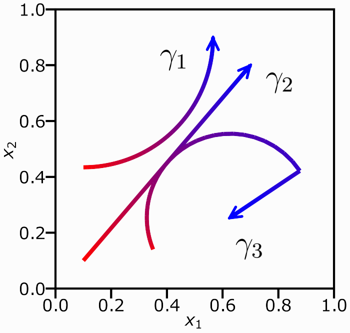
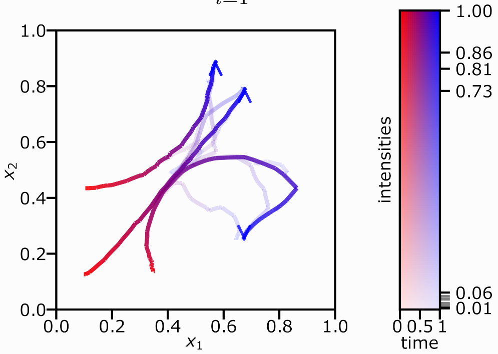
Algorithm: Generalized Frank-Wolfe $\rightsquigarrow \quad t \mapsto \mu_t^k = \sum_{i=1}^{M} \lambda_i \, \delta_{\gamma_i(t)}$



Ground Truth

$$\bar{\mu}_t = \delta_{\gamma_1(t)} + \delta_{\gamma_2(t)} + \delta_{\gamma_3(t)}$$



Reconstruction with data

$$y_t = K_t \mu_t + 20\% \text{ Noise}$$

[5] Bredies, Carioni, Fanzon, Romero. **Found. of Computational Mathematics** (2023)

Conclusion

1. Algorithm for computing **sparse** solutions to

$$\min_{u \in X} \; L(u) + R(u)$$

   in **Banach** space

2. **Linear** convergence when solution is **sparse** + **"source condition"**

3. General framework for **dynamic** inverse problems

4. Application to **Dynamic MRI**

Thank You!

References

Generalized Frank-Wolfe Algorithm

[1] Bredies, Carioni, **Fanzon**, Walter. **Mathematical Programming** (2023)

Particles Tracking + Dynamic Inverse Problems

[2] **Fanzon**, Bredies. **ESAIM: Mathematical Modelling and Numerical Analysis** (2020)

[3] Bredies, Carioni, **Fanzon**, Romero. **Bulletin London Mathematical Society** (2021)

[4] Bredies, Carioni, **Fanzon**. **Communications in PDEs** (2022)

[5] Bredies, Carioni, **Fanzon**, Romero. **Found. of Computational Mathematics** (2023)