Forces on a body with a vortex-dominated wake

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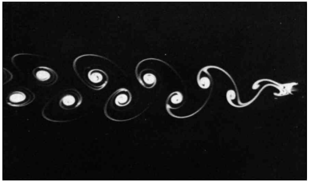
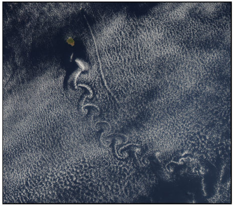
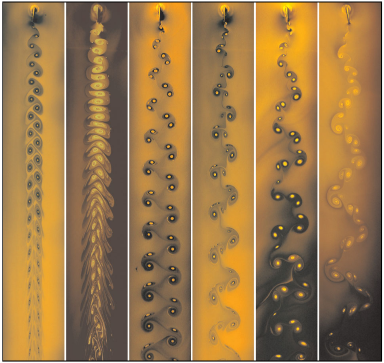
Wakes are often dominated by vortex structures



van Dyke

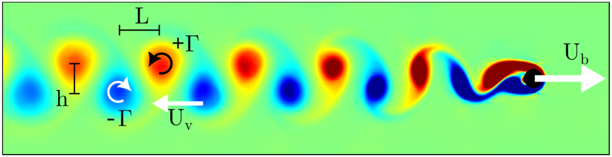


Leeward vortices in the atmosphere



Schnipper et. al (2009)

Kármán found a way to relate the wake vortex structure to forces on the body



numerical simulation of 2D flow past a circular cylinder at low Re

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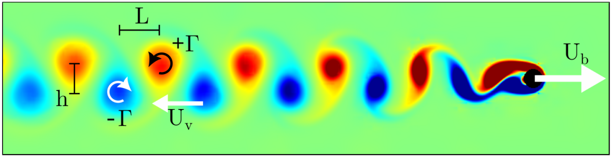

numerical simulation of 2D flow past a circular cylinder at low Re

$$F_D = \rho \Gamma \frac{h}{L} \left(U_b - 2U_v \right) + \frac{\rho \Gamma^2}{2\pi L} \tag{1}$$

Kármán (1911)

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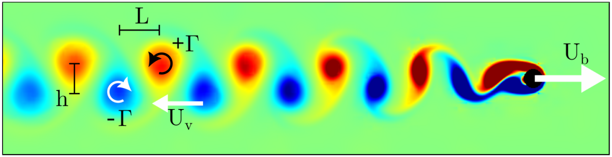

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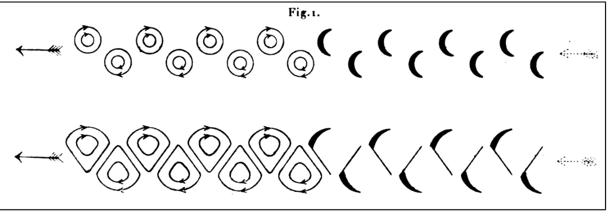
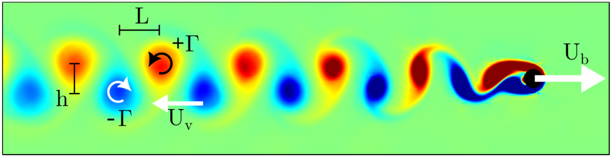


Henri Bénard, "Formation de centres de giration à l'arrière d'un obstacle en mouvement", Comptes Rendus hebdomadaires des Séances de l'Académie des Sciences 1908



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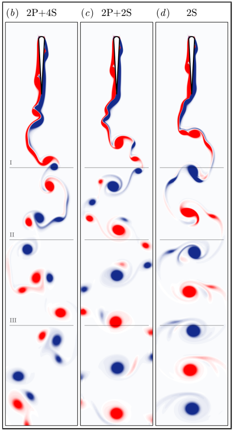
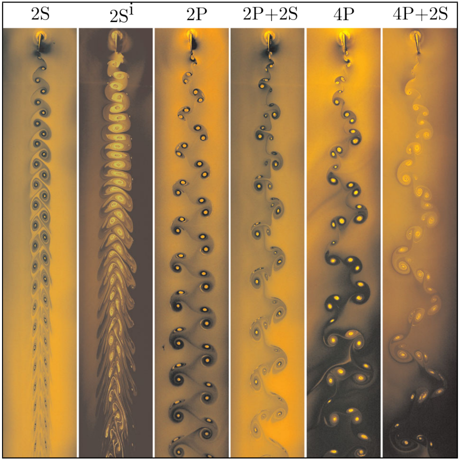
What assumptions were made by Kármán ?

- 2D flow
- Time-periodic flow
- Repeating vortex pattern
- $N = 2$ vortex street
- Horizontal-only motion
- Point vortices, ideal flow
- Vortices in relative equilibrium

Can we generalize Kármán 's drag law for 'exotic' vortex strets?



Colvert, Alsalman & Kanso (2018)



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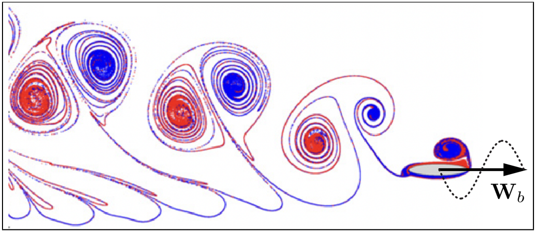
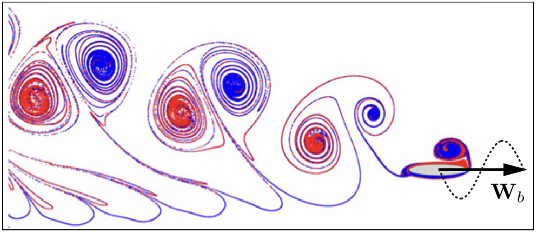


adapted from Kozlowski & Kudela (2014)

We assume:

- 2D flow
- Body moves steadily/periodically with an average velocity $\boldsymbol{W}_b = (U_b, V_b)$
- Equal positive / negative vorticity shed into the wake
- Wake consists of vortices shed along a common axis with possible inclination
- Vortices move collectively with an average velocity $\boldsymbol{W}_v = (U_v, V_v)$
- At this stage, point vortices need not be assumed



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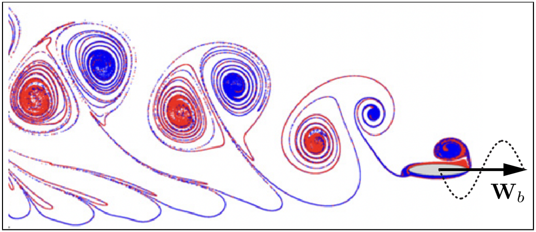

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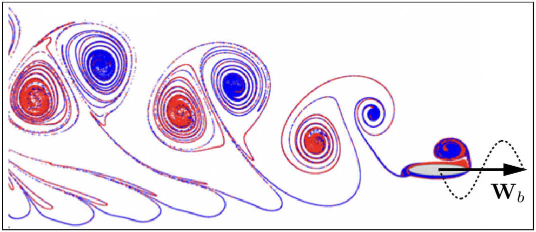

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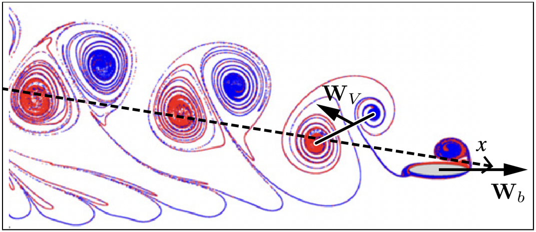

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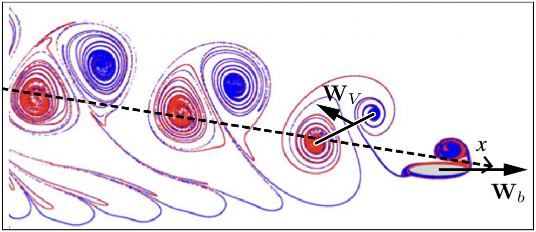

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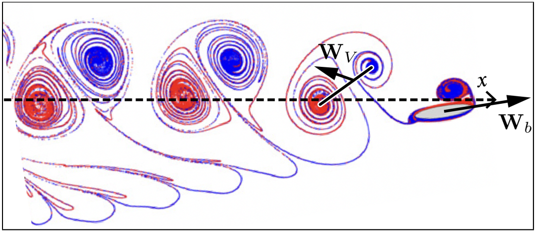


adapted from Kozlowski & Kudela (2014)

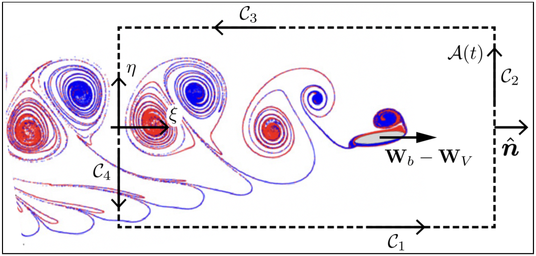
- Control volume moves with $\boldsymbol{W}_v$, the average speed of the vortices
- Define new variables

$$\boldsymbol{x}_r = (\xi, \eta) = \boldsymbol{x} - \boldsymbol{W}_v t \qquad \zeta = \xi + i\eta = z - W_v t$$

$$\boldsymbol{v}_r = (u_r, v_r) = \boldsymbol{v} - \boldsymbol{W}_v \qquad w_r = u_r - iv_r = w - W_v$$

- The body still has motion relative to the control volume
- New vortices are created over time in the interior

Consider a control volume moving with the vortices



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Newton's 2nd Law for the control volume (per unit depth into page)

$$\frac{D}{Dt}\int_{\text{sys}} \rho \boldsymbol{v} \mathrm{d}A = \frac{\partial}{\partial t}\int_{\mathcal{A}(t)} \rho \boldsymbol{v} \mathrm{d}A + \oint_{\mathcal{C}} \rho \boldsymbol{v}(\boldsymbol{v} - \boldsymbol{W}_v) \cdot \boldsymbol{n} \mathrm{d}s = \boldsymbol{f}(t) \tag{2}$$

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Rate of change of linear momentum $\mathbf{J}$ inside $\mathcal{A}$.

$$\mathbf{J}(t;\mathcal{A}) = (J_x, J_y) \equiv \int_{\mathcal{A}(t)} (\mathbf{v}_r + \mathbf{W}_V)\, \mathrm{d}A = \mathbf{J}_r(t;\mathcal{A}) + \mathbf{W}_V A$$

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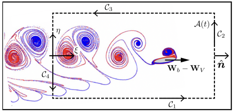

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$$\mathbf{J}_r(t;\mathcal{A}) = \underbrace{\int_{\mathcal{A}(t)} \mathbf{x}_r \times \boldsymbol{\omega}_r\,\mathrm{d}A}_{\text{vortical impulse } \mathbf{I}_{\mathcal{A}}} \underbrace{- \oint_{\mathcal{C}} [\mathbf{x}_r \times (\mathbf{n} \times \mathbf{v}_r)]\,\mathrm{d}s}_{\text{potential impulse } \mathbf{I}_{\mathcal{C}}}$$

$\mathbf{J}_r$ is the sum of a vortical impulse and a potential impulse — Wu et.al (2015)

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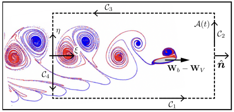

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In complex form,

$$\mathbf{I}_{\mathcal{A}}(t;\mathcal{A}) = H_{rx} + \mathrm{i}H_{ry} = -\mathrm{i}\int_{\mathcal{A}(t)} \zeta \boldsymbol{\omega}_r \mathrm{d}A$$

Flow in $\mathcal{A}$ is allowed to be rotational.

$$\mathbf{I}_{\mathcal{C}}(t;\mathcal{C}) = \underbrace{-\mathrm{i}\oint_{\mathcal{C}} \phi_r\,\mathrm{d}\zeta}_{\text{Let } \partial_t \text{ of this be } (*)}$$

Flow on $\mathcal{C}$ is irrotational.

Newton's 2nd Law for the control volume (per unit depth into page)

$$\frac{D}{Dt}\int_{\text{sys}}\rho\boldsymbol{v}\,\mathrm{d}A = \frac{\partial}{\partial t}\int_{\mathcal{A}(t)}\rho\boldsymbol{v}\,\mathrm{d}A + \oint_{\mathcal{C}}\rho\boldsymbol{v}(\boldsymbol{v}-\boldsymbol{W}_v)\cdot\boldsymbol{n}\,\mathrm{d}s = \boldsymbol{f}(t) \tag{2}$$

Flux of linear momentum $\mathbf{J}$ across the contour $\mathcal{C}$

$$\oint_{\mathcal{C}}\rho\mathbf{v}\left(\mathbf{v}-\mathbf{W}_V\right)\cdot\mathbf{n}\,\mathrm{d}s = \rho\oint_{\mathcal{C}}\mathbf{v}_r\left(\mathbf{v}_r\cdot\mathbf{n}\right)\mathrm{d}s + \rho\mathbf{W}_V\int_{\mathcal{A}(t)}\left(\nabla_r\cdot\mathbf{v}_r\right)\mathrm{d}A$$

$$= \rho\oint\mathbf{v}_r\,\mathrm{d}\psi_r, \quad \text{where } u_r = \frac{\partial\psi_r}{\partial\eta}, \quad v_r = -\frac{\partial\psi_r}{\partial\xi}$$

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In complex form,

$$\oint_{\mathcal{C}} \rho \boldsymbol{v}(\boldsymbol{v} - \boldsymbol{W}_v) \cdot \boldsymbol{n} \mathrm{d}s = \underbrace{\rho \oint_{\mathcal{C}} \overline{w_r} \mathrm{d}\psi_r}_{(**)}$$

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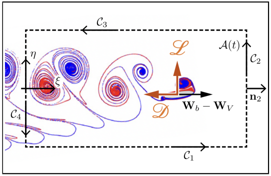

Instantaneous external force on $\mathcal{A}$ in complex form:

$$\boldsymbol{f}(t) = f_x(t) + i f_y(t) = \mathcal{D}(t) - \mathrm{i}\mathcal{L}(t) + \mathrm{i}\oint_{\mathcal{C}} p \mathrm{d}\zeta$$

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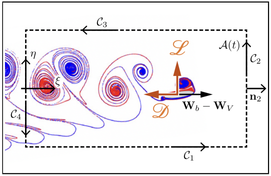

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$$= \underbrace{\qquad\qquad\qquad}_{(*)} + \oint_{\mathcal{C}} \left(\overline{w_r + W_v}\right)^2 \mathrm{d}\bar{\zeta} + \underbrace{\rho\oint_{\mathcal{C}} \overline{w_r}\,\mathrm{d}\psi_r}_{(**)}$$

unsteady form of Bernoulli's equation.

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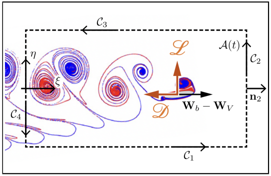

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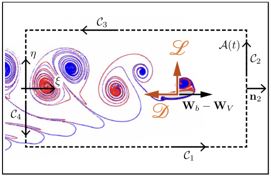
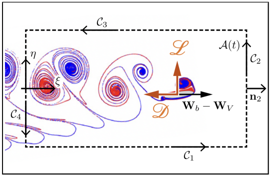
unsteady form of Bernoulli's equation.

Terms (*) and (**) appear on both sides of Newton's 2nd Law

Putting it all together

- Take the complex conjugate of Newton's 2nd Law
- Make $\mathcal{A}$ so large that $w_r = W_v$ on $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ (but not on $\mathcal{C}_4$)





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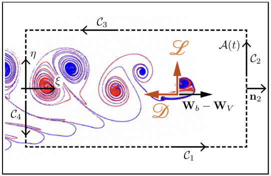

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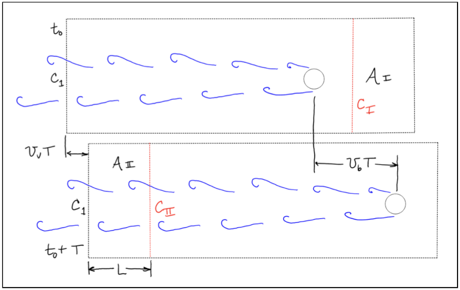
Integrate over period and divide by T to find the average forces

$$\mathcal{D} + \mathrm{i}\,\mathcal{L} = \frac{1}{T}\int_{t_0}^{t_0+T} \hat{\sigma}(t)\,\mathrm{d}t + \frac{\rho}{T}\underbrace{\overline{\Delta \mathbf{I}_{\mathcal{A}}}}_{\substack{\text{change in vortical impulse} \\ \text{inside } \mathcal{A} \text{ over one period}}}$$

Q. How much does the vortical impulse change over one period?

$$\mathcal{D} + \mathrm{i}\,\mathcal{L} = \frac{1}{T}\int_{t_0}^{t_0+T} \hat{\sigma}(t)\,\mathrm{d}t + \frac{\rho}{T}\,\overline{\Delta \mathbf{I}_{\mathcal{A}}}$$



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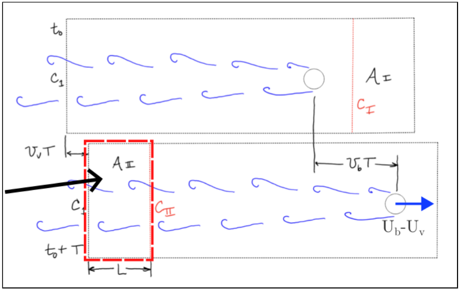
Answer: equal to the vortical impulse in here!



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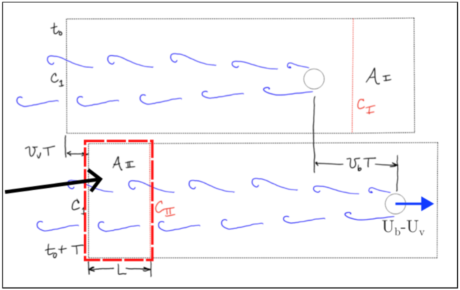

Answer: equal to the vortical impulse in here!

Also, the period $T = \dfrac{L}{U_b - U_v}$, hence we get

$$\mathcal{D} + \mathrm{i}\,\mathcal{L} = \frac{1}{T}\int_{t_0}^{t_0+T} \hat{\sigma}(t)\,\mathrm{d}t + \rho\left(\frac{U_b - U_v}{L}\right)\overline{\mathbf{I}_{\mathcal{A}_{II}}}$$

where $\mathbf{I}_{\mathcal{A}_{II}} = -\mathrm{i}\displaystyle\int_{\mathcal{A}_{II}} \zeta\omega_r \mathrm{d}A$

Assuming a point-vortex description of the wake ...

For point vortices, $\mathbf{I}_{\mathcal{A}_{II}} = P - iQ$

$$\mathcal{D} + \mathrm{i}\,\mathcal{L} = \rho\left(\frac{U_b - U_V}{L}\right)(P + \mathrm{i}Q) + \frac{\rho}{4\pi L}\left\{\sum_{j=1}^{N}\Gamma_j^2 + \underbrace{\sum_{j=1} N \sum_{k=1}^{N}\Gamma_j\,\Gamma_k\left[\frac{\pi}{L}(z_j - z_k)\right]\ \cot\left[\frac{\pi}{L}(z_j - z_k)\right]}_{\text{simplifies to } -4\pi\mathbf{I}_{\mathcal{A}_{II}}U_v}\right\}$$

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if the vortices are in relative equilibrium

Also valid for finite-area vortices —
Saffman & Schatzman (1982) followed by O'Neil (2009)
have shown this for $N = 2$

We obtain a generalized Kármán-like drag law for N-vortex streets

$$\mathcal{D} + \mathrm{i}\,\mathcal{L} = \rho \left(\frac{U_b - 2U_V}{L} \right) P + \frac{\rho}{4\pi L} \sum_{j=1}^{N} \Gamma_j^2 + i \left(\quad \ldots \quad \right) \qquad (3)$$

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Summary & Conclusions

- Given some assumptions about periodic vortex streets with N vortices per period:
- The forces are found to depend on the vortical impulse in one period of the wake and the pressure on the boundary $\mathcal{C}_4$
- Forces depend only on the vortical impulse, self-induced speed of the vortices, and the sum of the strengths squared
- Relative equilibria of $N > 2$ vortices on singly-periodic domain are candidates for models of 'exotic' wakes
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> Finally, Kármán's ultimate objective with his vortex wake model was to establish a formula for the drag on the bluff body producing the wake. In this he succeeded with what we today know as the *Kármán drag law*. The vortex-street patterns with three vortices per period, e.g., those illustrated in Fig. 8, should also yield a drag law much like von Kármán's. However, it may be possible to extend this further and derive a drag law even for cases where the three-vortex-per-cycle wake is evolving downstream. We leave this as an open problem to which we intend to return.

Aref, Stremler & Ponta J. Fluid Struct. (2006)

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