The periodic N-vortex ring problem

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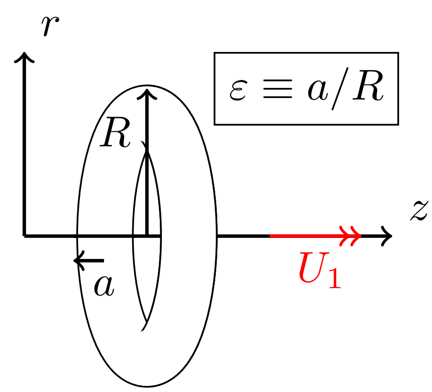
Vorticity Dynamics in Classical and Quantum Fluids
SIAM Conference on Nonlinear Waves and Coherent Structures
Baltimore, MD

June 24, 2024

Vortex rings are solutions to the Euler equations in axisymmetric form

$$\frac{\partial \omega}{\partial t} + \frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial \omega}{\partial r} - \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial \omega}{\partial z} = -\frac{1}{r^2}\frac{\partial \psi}{\partial r}\omega\frac{\partial \omega}{\partial r}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 \psi}{\partial z^2} = -\omega$$



- Scalar-valued ω confined to a torus shape with small ε
- ψ arises from the Green's function for the Laplacian in cylindrical coordinates
- The ring moves forward with speed

$$U_1(\varepsilon) \approx \frac{1}{4\pi}\left(\log\frac{8}{\varepsilon} - \frac{1}{4}\right)$$

$B^2 = Ra^2 =$ const.

Thomson. "The translatory velocity of a circular vortex ring". *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 33 (1867)

Successive vortex rings are often used to model wakes

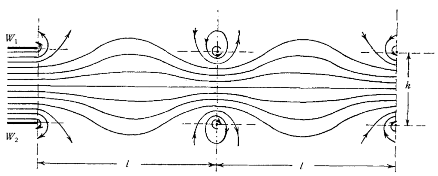
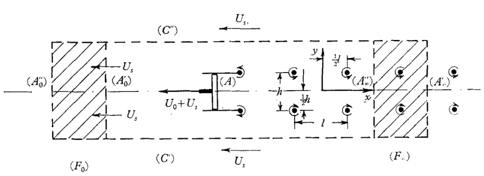


FIGURE 7. Symmetric vortex trail behind a jet propeller.



Siekmann 1963

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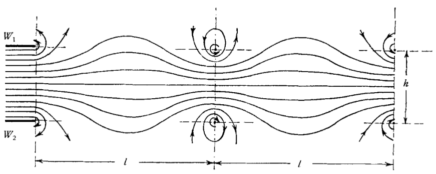
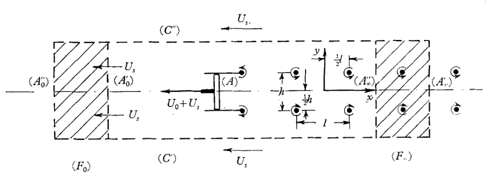


FIGURE 7. Symmetric vortex trail behind a jet propeller.



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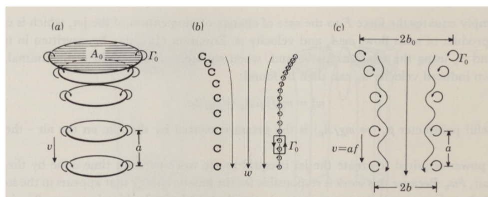

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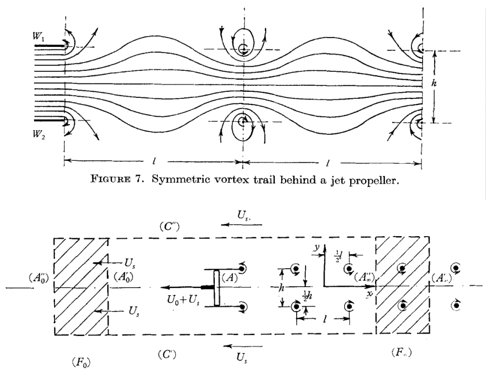

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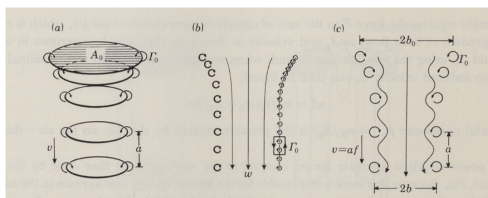

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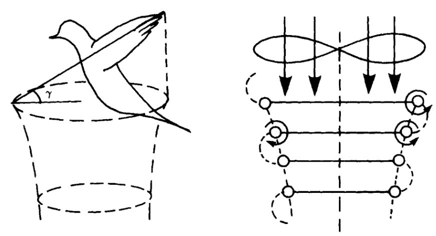

Rayner 1979

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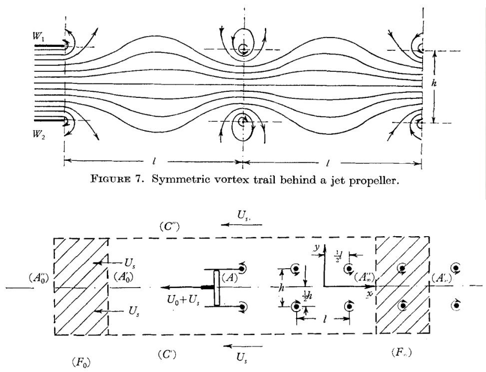

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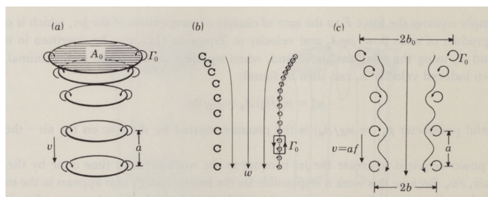

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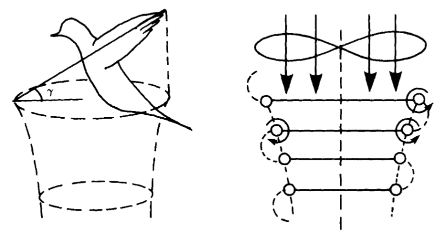

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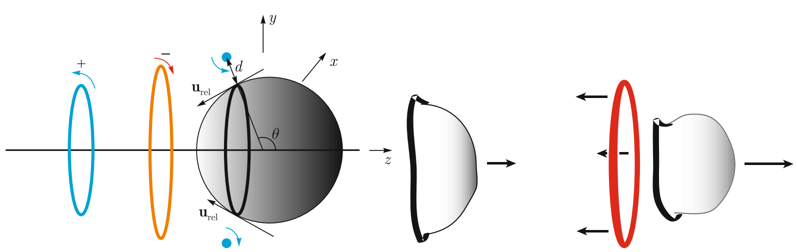

Tallapragada2013

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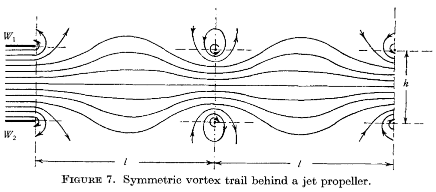

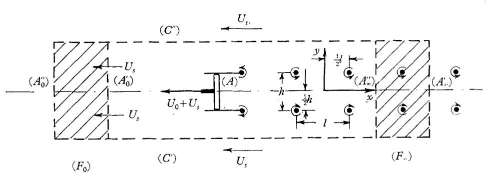

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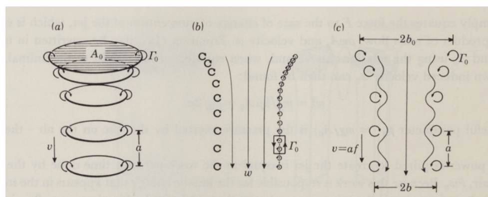

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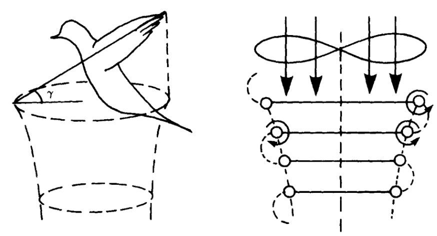

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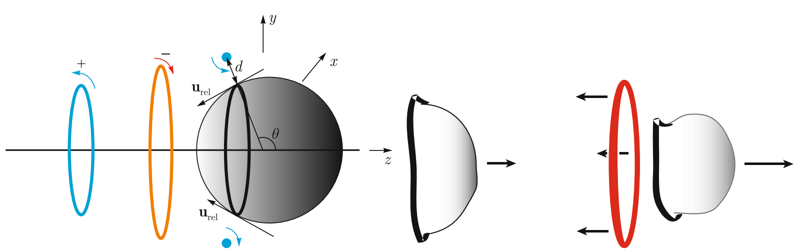

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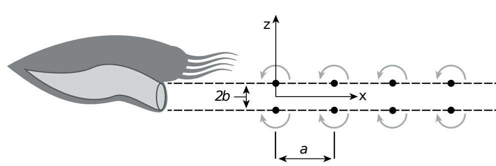
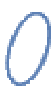


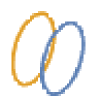
Gordon, Blickhan, Dabiri, and Videler 2017

It is well known how two vortex rings interact

'Pass through'

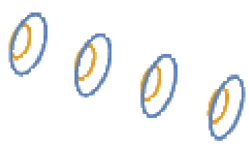


'Leapfrogging'



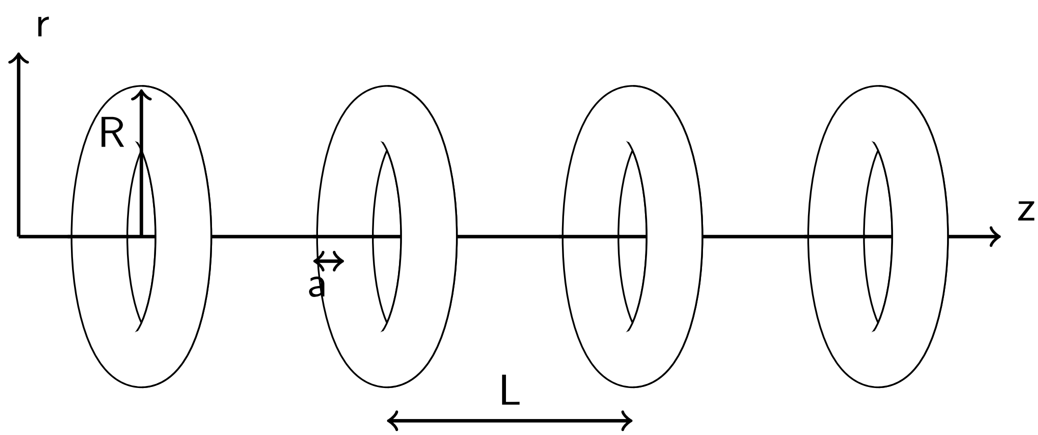
delineation of these regimes is due to Lord Kelvin.

How do two *arrays* of vortex rings interact?



what other regimes might exist?

An infinite coaxial array of identical vortex rings



$$\varepsilon \equiv \frac{a}{R}, \quad \lambda \equiv \frac{L}{R}$$

Vortex rings induce two streamline topologies

$$\psi(z,r;\varepsilon) = \overbrace{\tilde{\psi}(z,r)}^{\text{time-dependent}} - \frac{1}{2}\overbrace{U_1(\varepsilon)}^{\text{self-induced speed}} r^2$$

- Thin annulus-shaped cloud at low ε
- Thick biconcave or elliptical cloud at high ε
- Critical $\varepsilon_c \approx 0.0116$, due to Hicks 1919

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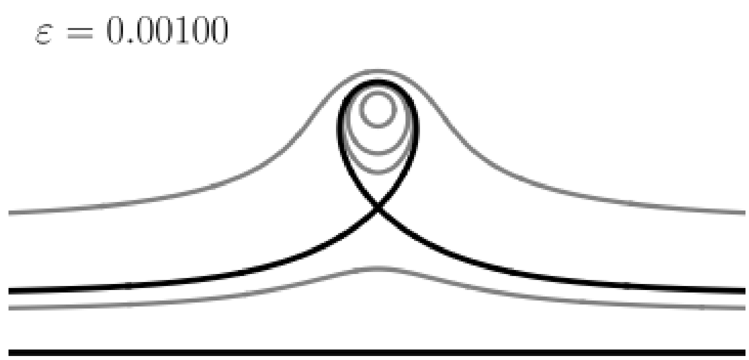

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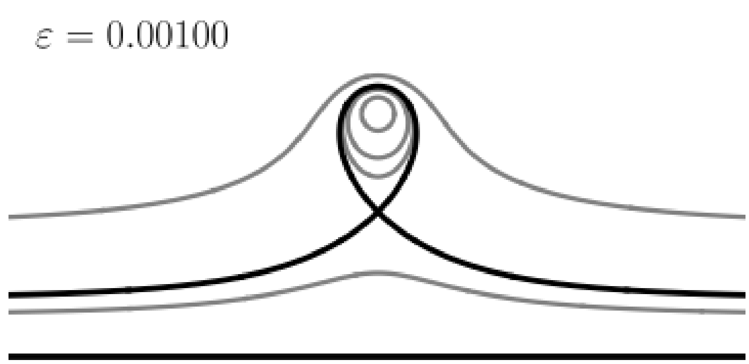
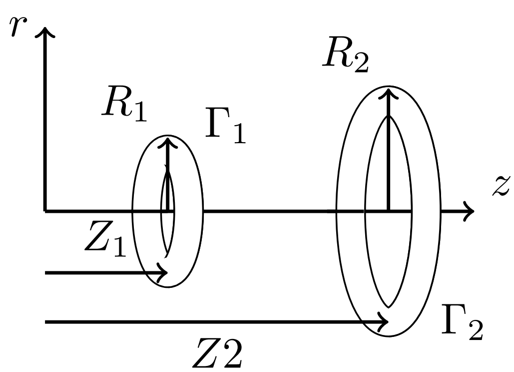


Adding additional vortex rings $\implies$ no privileged co-moving frame ...



$$Z_1(t), Z_2(t), R_1(t), R_2(t)$$

- Different regimes of inter-vortex motion are well-known (Helmholtz 1858)
- A recent comprehensive treatment is Borisov, Kilin, and Mamaev. "The dynamics of vortex rings: Leapfrogging, choreographies and the stability problem". *Regular and Chaotic Dynamics* 18.1 (11, 2013)
- Note the absence of any 'scale' for Γ's, R's and Z's — 'everything is relative'

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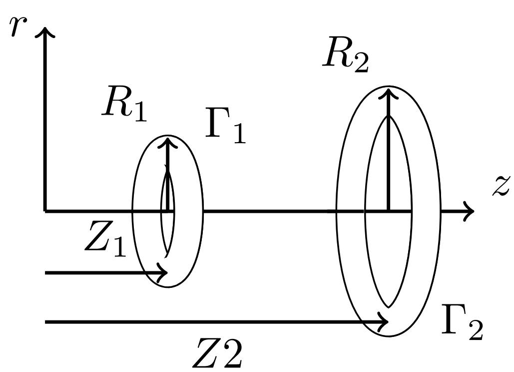

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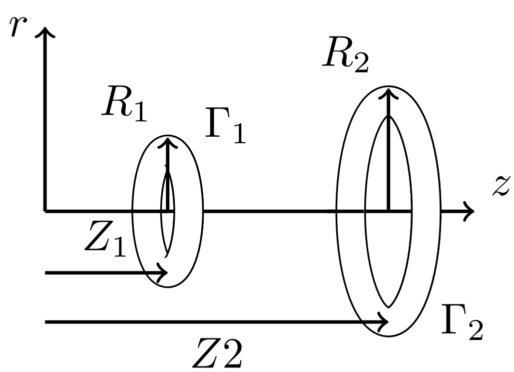

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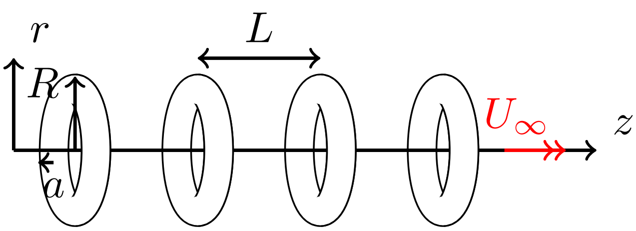
... except for the special case: an infinite array of identical rings



- *Also* move without change of shape, with speed U_∞

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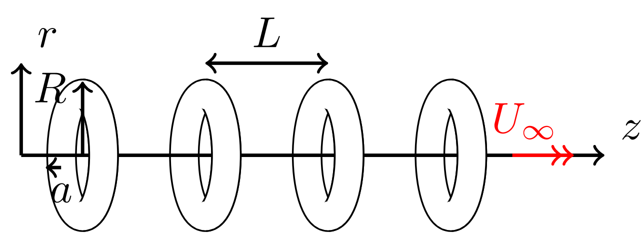

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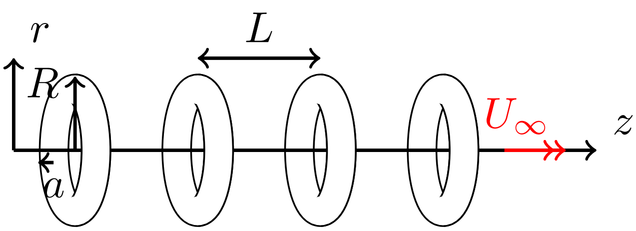
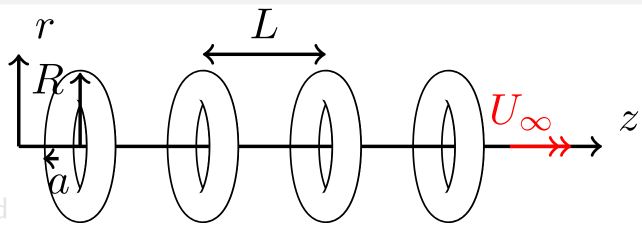


- *Also* move without change of shape, with speed U_∞
- Independently discovered by Vasilev 1916 and Levy and Forsdyke 1927.
- Parameterized by two non-dimensional numbers

$$\varepsilon \equiv a/R, \quad \lambda \equiv L/R$$

This solution to the Euler equation uses a periodic Green's function and a modified self-induced speed U_∞



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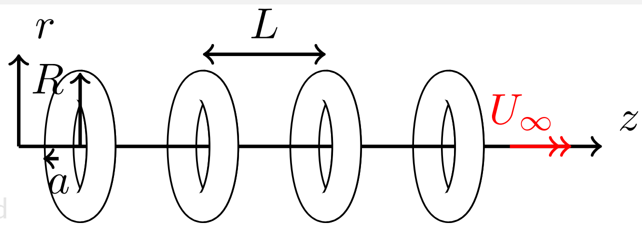

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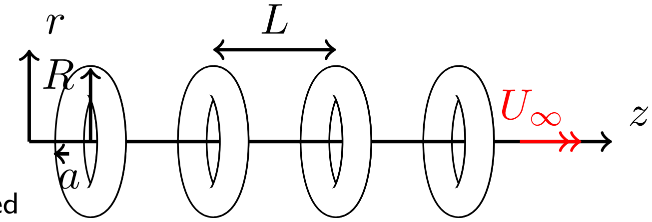

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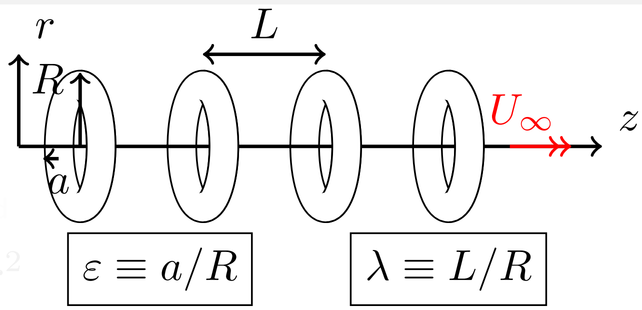

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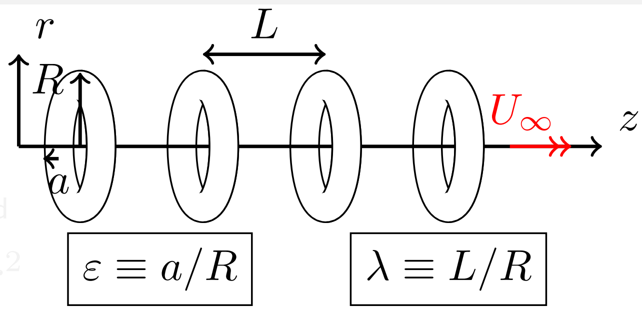

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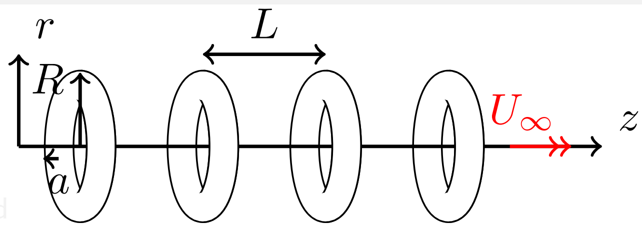

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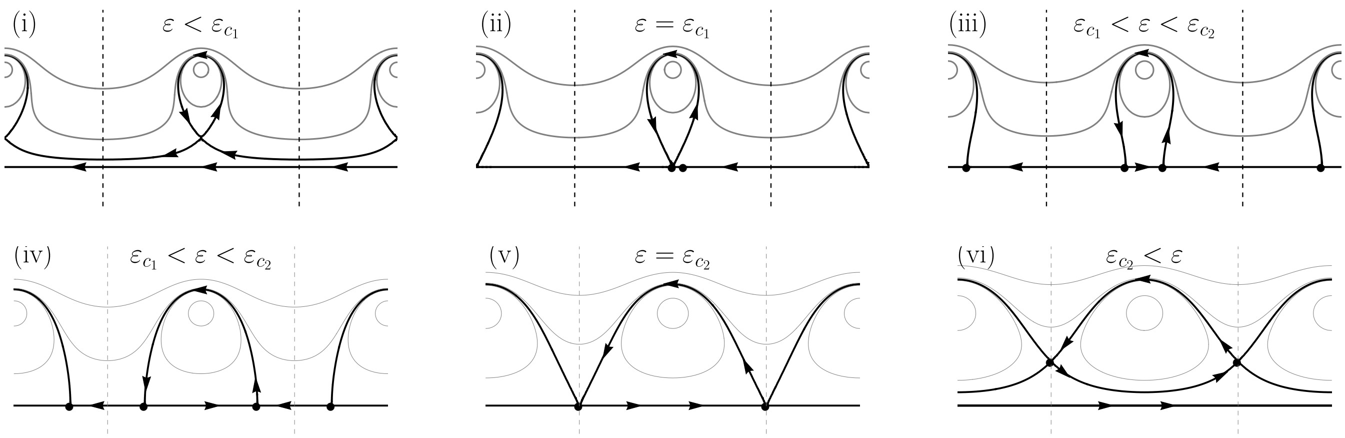
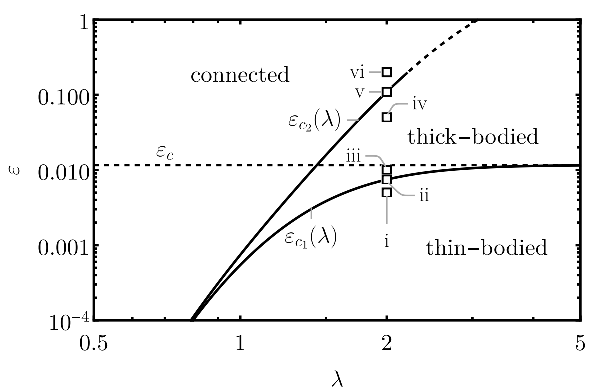
An array of vortex rings exhibits three distinct streamline topologies





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Masroor and Stremler 2022

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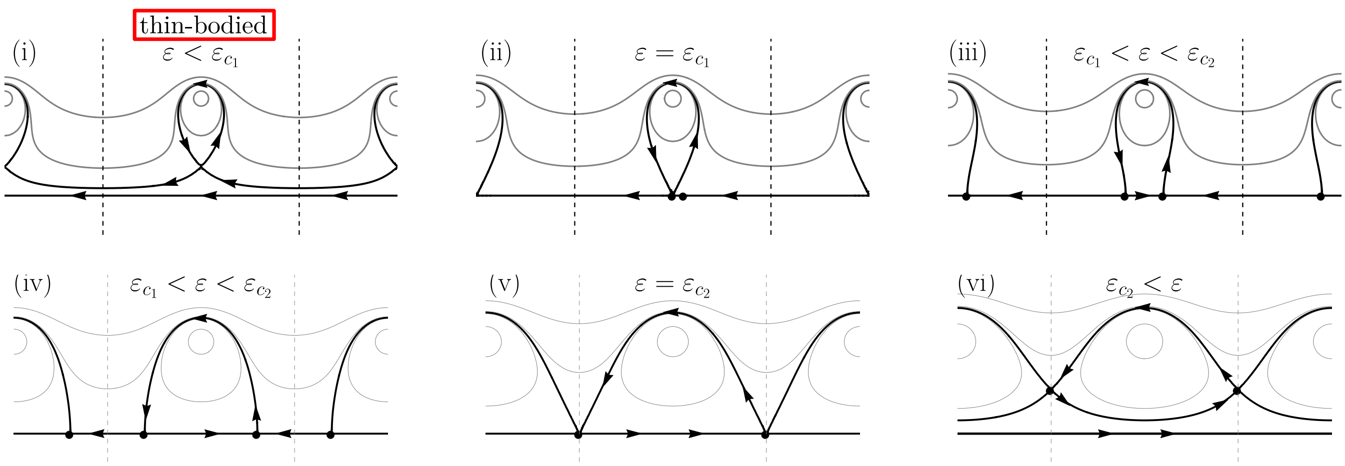

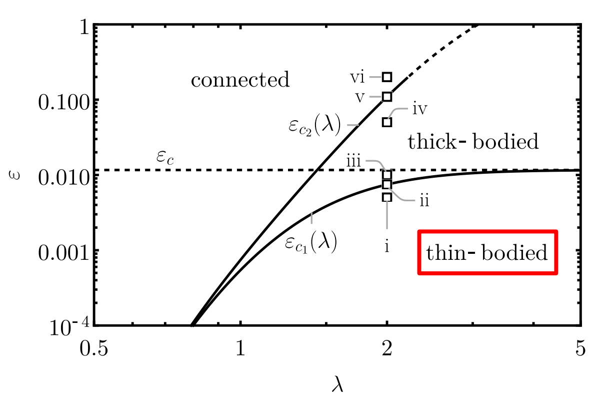

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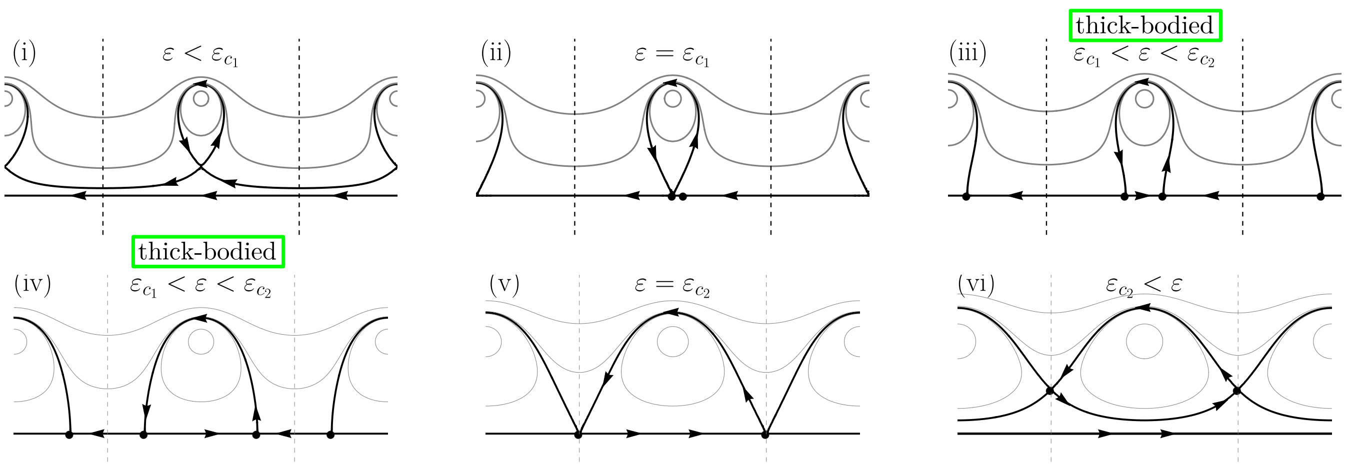

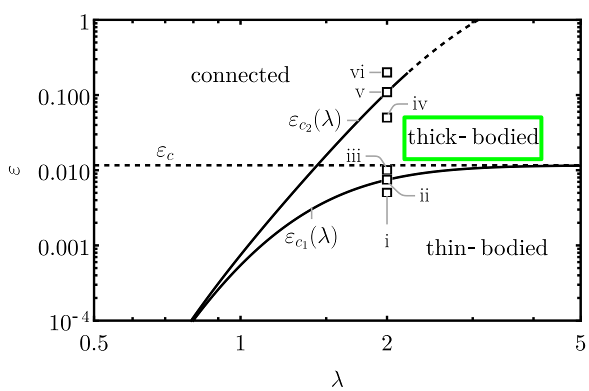

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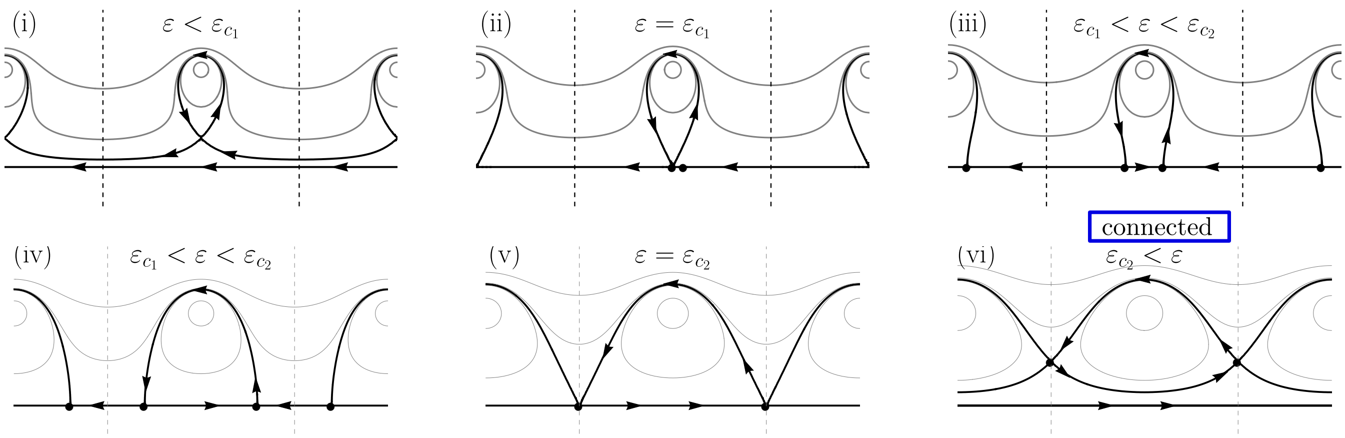

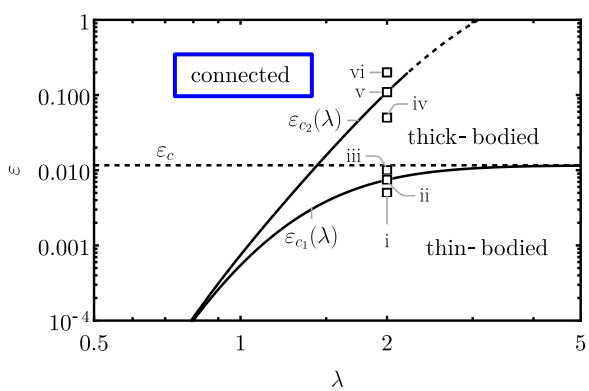

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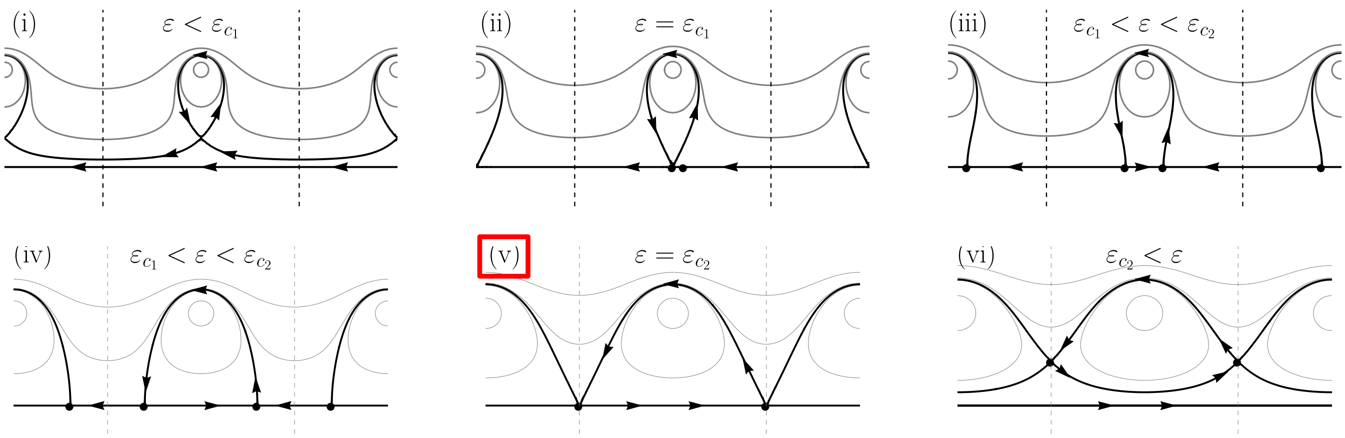

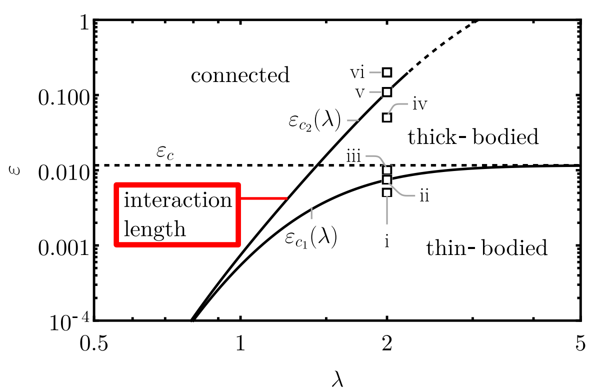

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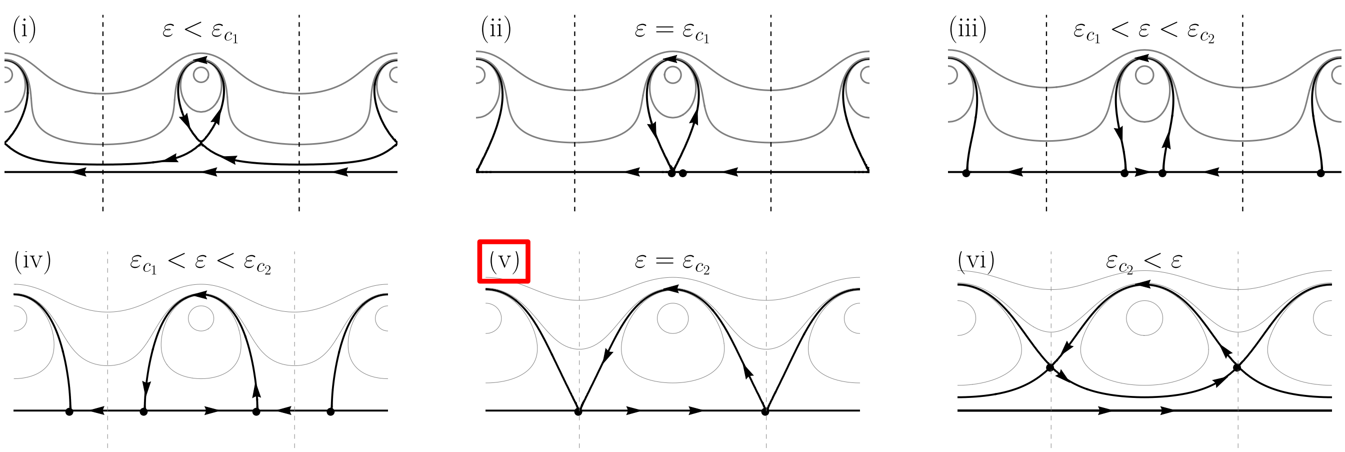
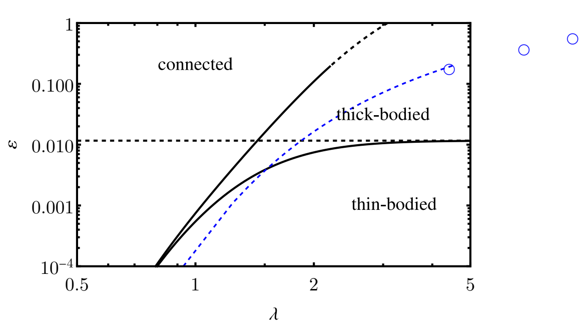


Optimal vortex formation length from **Gharib1998**



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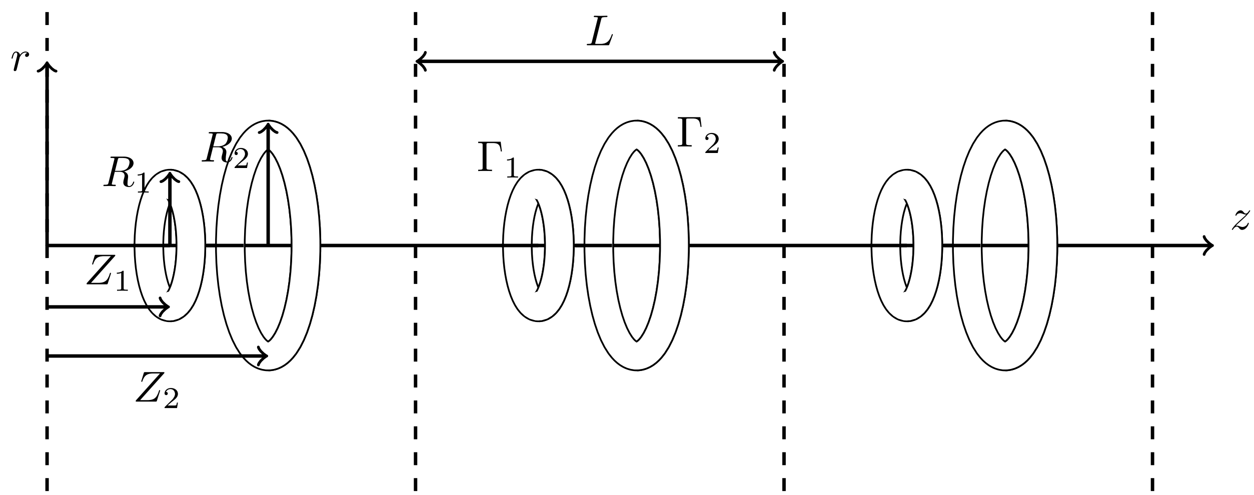
Interacting Vortex Ring Arrays



The interaction of N coaxial vortex rings can be modeled as a dynamical system with N degrees of freedom in $2N$-dimensional phase space:

$$\dot{Z}_i = \frac{1}{\Gamma_i R_i}\frac{\partial H}{\partial R_i}, \quad \dot{R}_i = -\frac{1}{\Gamma_i R_i}\frac{\partial H}{\partial Z_i}$$

whose Hamiltonian H has a self-interaction term and a Green's function:

$$H = \frac{1}{2\pi}\sum_{i=1}^{N}\Gamma_i^2 R_i\left(\log\frac{8R_i^{3/2}}{B_i} - \frac{7}{4}\right) + \sum_{i\neq j}^{N}\Gamma_i\Gamma_j \underbrace{G(R_i, Z_i; R_j, Z_j)}_{\psi \text{ due to ring } j \text{ evaluated at ring } i}$$

The existence of another integral in involution

$$P \equiv \sum_{j}^{N}\Gamma_j R_j^2; \quad \frac{dP}{dt} = 0$$

guarantees a canonical transformation to conjugate variables

$$\dot{\xi}_j = \{\xi_j, H\}$$
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with Hamiltonian

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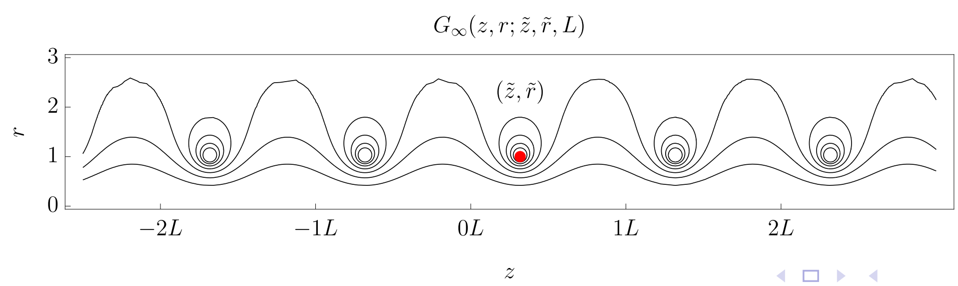

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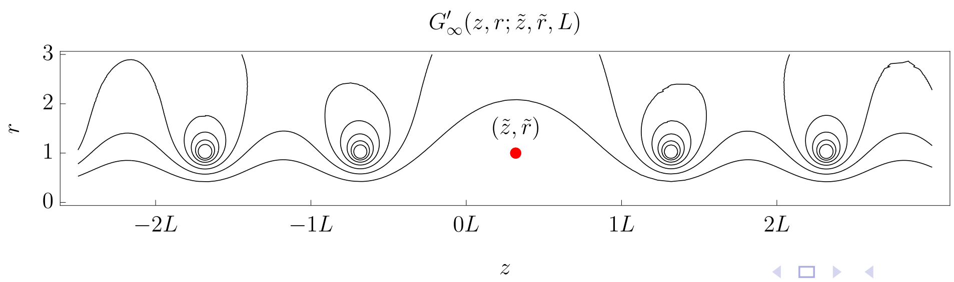

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Interpreting the Hamiltonian for 2 interacting vortex ring arrays

$$H = \frac{1}{2\pi}\sum_{i=1}^{N}\Gamma_i^2 R_i\left(\log\frac{8R_i^{3/2}}{B_i} - \frac{7}{4}\right) + \sum_{i\neq j}\Gamma_i\Gamma_j G_\infty(R_i, Z_i; R_j, Z_j, L) + \sum_i \Gamma_i^2 G'_\infty(R_i, Z_i; R_i, Z_i, L)$$

Let $N = 2$

$H = \frac{1}{2\pi}\Gamma_1^2 R_i\left(\log\frac{8R_1^{3/2}}{B_i} - \frac{7}{4}\right)$	Self-induction of ring 1
$+\frac{1}{2\pi}\Gamma_2^2 R_1\left(\log\frac{8R_2^{3/2}}{B_i} - \frac{7}{4}\right)$	Self-induction of ring 2
$+\Gamma_1\Gamma_2 G_\infty(R_1, Z_1; R_2, Z_2, L)$	Effect on ring 1 induced by ring 2 and its images
$+\Gamma_2\Gamma_1 G_\infty(R_2, Z_2; R_1, Z_1, L)$	Effect on ring 2 induced by ring 1 and its images
$+\Gamma_1^2 G'_\infty(R_1, Z_1; R_1, Z_1, L)$	Effect on ring 1 induced by its own images
$+\Gamma_2^2 G'_\infty(R_2, Z_2; R_2, Z_2, L)$	Effect on ring 2 induced by its own images

Interpreting the reduced variables ξ and η

For N coaxial vortex ring (array)s

- 'Real space' $H(\boldsymbol{Z}, \boldsymbol{R}) \quad (Z_1, ...Z_N, R_1, ..., R_N)$
- 'Phase space' $H(\boldsymbol{\xi}, \boldsymbol{\eta}) \quad (\xi_1, \xi_2, ...\xi_{N-1}, \eta_1, \eta_2, ...\eta_{N-1})$

$$\xi_0 = \sum_{i=1}^{N} \Gamma_i Z_i \Bigg/ \sum_{i=1}^{N} \Gamma_i = \frac{\Gamma_1 Z_1 + \Gamma_2 Z_2}{\Gamma_1 + \Gamma_2}$$

$$\xi_i = Z_{i+1} - \sum_{j=1}^{i} \Gamma_j Z_j \Bigg/ \sum_{j=1}^{i} \Gamma_j = Z_2 - Z_1$$

$$\eta_i = \left(R_{i+1}^2 - \sum_{j=1}^{i} \Gamma_j R_j^2 \Bigg/ \sum_{j=1}^{i} \Gamma_j \right) \Gamma_{i+1} \sum_{j=1}^{i} \Gamma_j \Bigg/ \sum_{j=1}^{i+1} \Gamma_j = (R_2^2 - R_1^2) \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2}$$

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Interpreting the reduced variables ξ and η for 2 vortex ring arrays

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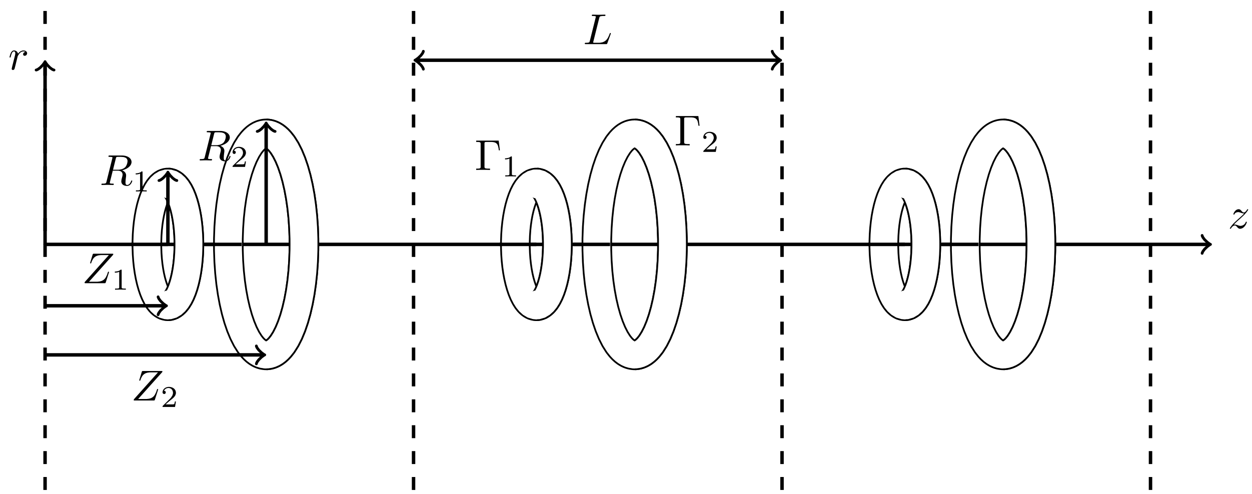
Non-dimensionalizing the problem

The spatial periodicity introduces a new length scale L ...



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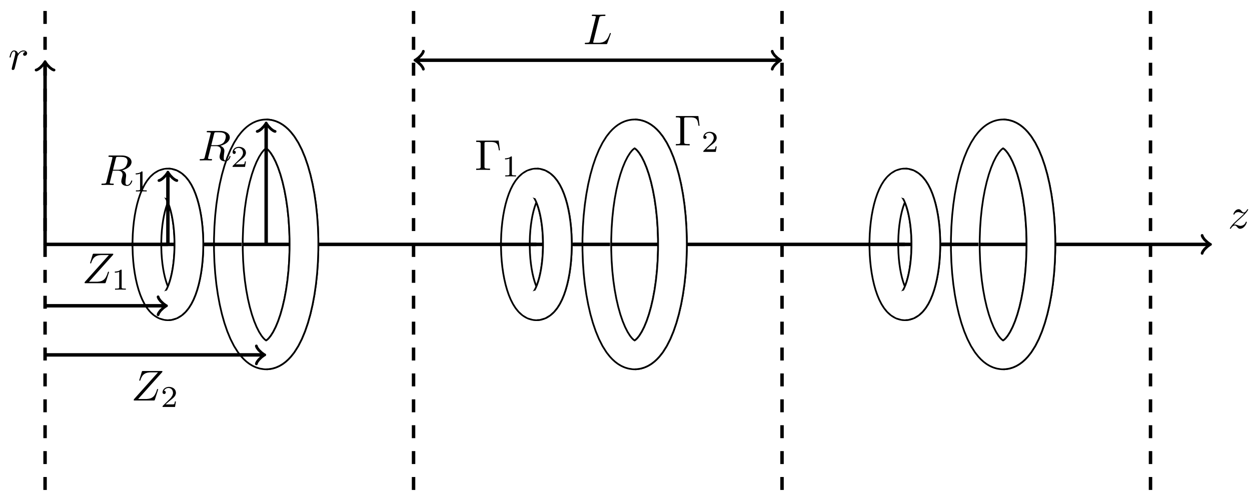

... motivating a re-scaling of all lengths by L.

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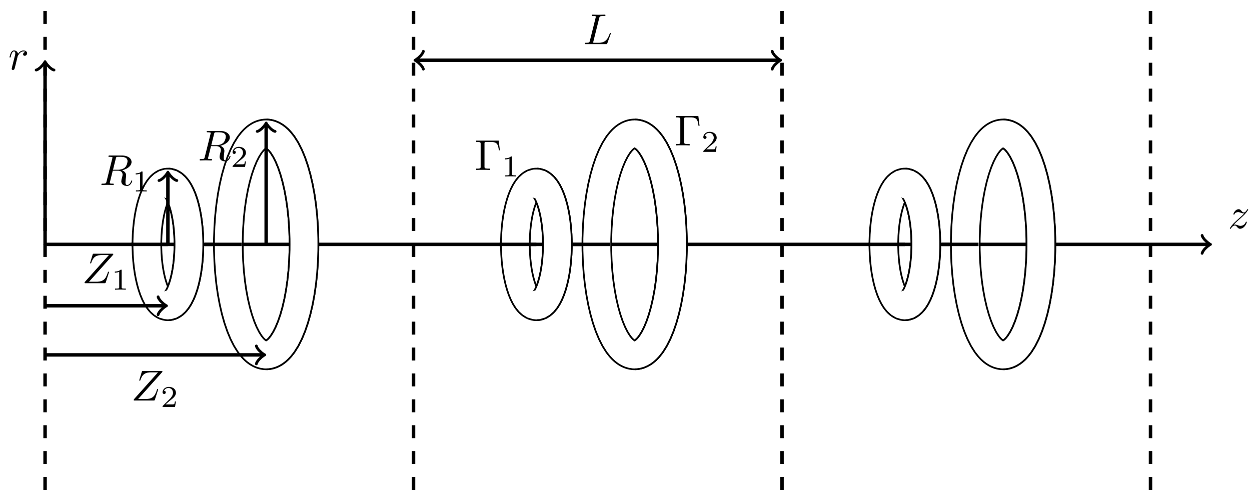

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also, recall $B^2 = Ra^2$

$$R_i^* \equiv \frac{R_i}{L}, \quad Z_i^* \equiv \frac{Z_i}{L}$$

$$B_i^* \equiv \frac{B_i}{L^{3/2}}$$

Re-scaled canonically conjugate variables ξ_j^* and η_j^*

We now have an N-degree of freedom non-canonical Hamiltonian system

$$\dot{Z}_j^* = \{Z_j^*, H^*\} = \frac{1}{\Gamma_j R_j^*}\frac{\partial H^*}{\partial R_j^*}, \quad \dot{R}_j^* = \{R_j^*, H^*\} = -\frac{1}{\Gamma_j R_j^*}\frac{\partial H^*}{\partial Z_j^*}$$

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Reduces to $(N-1)$-degree of freedom *canonical* Hamiltonian system:

$$\dot{\xi}_j^* = \{\xi_j^*, H^*\} = \frac{\partial H^*}{\partial \eta_j^*} \qquad \text{where } P^* \equiv \frac{P}{\Gamma_1 L^2}$$

$$\dot{\eta}_j^* = \{\eta_j^*, H^*\} = -\frac{\partial H^*}{\partial \xi_j^*} \qquad \text{and } \{P^*, H^*\} = 0$$

$$\dot{\xi}_0^* = \frac{\partial H^*}{\partial P^*}$$

Re-scaled canonically conjugate variables ξ_j^* and η_j^*

We now have an N-degree of freedom non-canonical Hamiltonian system

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Choose to re-scale all vortex strengths by Γ_1.

Reduces to $(N-1)$-degree of freedom *canonical* Hamiltonian system:

$$\dot{\xi}_j^* = \{\xi_j^*, H^*\} = \frac{\partial H^*}{\partial \eta_j^*} \qquad \text{where } P^* \equiv \frac{P}{\Gamma_1 L^2}$$

$$\dot{\eta}_j^* = \{\eta_j^*, H^*\} = -\frac{\partial H^*}{\partial \xi_j^*} \qquad \text{and } \{P^*, H^*\} = 0$$

$$\dot{\xi}_0^* = \frac{\partial H^*}{\partial P^*}$$

Re-scaled canonically conjugate variables ξ_j^* and η_j^*

We now have an N-degree of freedom non-canonical Hamiltonian system

$$\dot{Z}_j^* = \{Z_j^*, H^*\} = \frac{1}{\Gamma_j R_j^*}\frac{\partial H^*}{\partial R_j^*}, \quad \dot{R}_j^* = \{R_j^*, H^*\} = -\frac{1}{\Gamma_j R_j^*}\frac{\partial H^*}{\partial Z_j^*}$$

with $H^* \equiv \dfrac{H}{\Gamma_1 L^2}$ and $G^* \equiv \dfrac{G}{L}$

Choose to re-scale all vortex strengths by Γ_1.

Reduces to $(N-1)$-degree of freedom *canonical* Hamiltonian system:

$$\dot{\xi}_j^* = \{\xi_j^*, H^*\} = \frac{\partial H^*}{\partial \eta_j^*} \qquad \text{where } P^* \equiv \frac{P}{\Gamma_1 L^2} \qquad \xi_j^* \equiv \frac{\xi_j}{L}$$

$$\dot{\eta}_j^* = \{\eta_j^*, H^*\} = -\frac{\partial H^*}{\partial \xi_j^*} \qquad \text{and } \{P^*, H^*\} = 0 \qquad \eta_j^* \equiv \frac{\eta_j}{\Gamma_1 L^2}$$

$$\dot{\xi}_0^* = \frac{\partial H^*}{\partial P^*}$$

Keeping count of our variables and parameters for general N

N vortex rings in a periodic domain
$(Z_1, Z_2, ...Z_N, R_1, R_2, ..., R_N)$

N-dof non-canonical Hamiltonian
$2N$ coordinates

Transform to
$(\xi_1, \xi_2, ...\xi_{N-1}, \eta_1, \eta_2, ..., \eta_{N-1})$

$(N-1)$-dof canonical Hamiltonian
$2(N-1)$ coordinates

2 'global' constants

- Impulse P^* — how do the strengths Γ and sizes R of the vortex rings compare with L ?
- B_1^* — how do the core sizes compare with the length L ?

$N-1$ parameters

- $\gamma_{j-1} \equiv \dfrac{\Gamma_j}{\Gamma_1}$ — relative strength of vortex ring j
- $\beta_{j-1} \equiv \dfrac{B_j}{B_1}$ — relative core thickness of vortex ring j

Keeping count of our variables and parameters for general N

N vortex rings in a periodic domain $(Z_1, Z_2, ...Z_N, R_1, R_2, ..., R_N)$ — N-dof non-canonical Hamiltonian, $2N$ coordinates

Transform to $(\xi_1, \xi_2, ...\xi_{N-1}, \eta_1, \eta_2, ..., \eta_{N-1})$ — $(N-1)$-dof canonical Hamiltonian, $2(N-1)$ coordinates

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The periodic 2-vortex ring problem in canonical form

Hamiltonian $H(\xi, \eta; P^*, B_1^*, \beta, \gamma)$

4-parameter family of 1-dof systems

Equations of motion:

$$\dot{\xi} = \{\xi, H\} = \frac{\partial H}{\partial \eta}, \quad \dot{\eta} = \{\eta, H\} = -\frac{\partial H}{\partial \xi}$$

with the usual Poisson bracket

$$\{f, g\} = \left(\frac{\partial f}{\partial \xi} \frac{\partial g}{\partial \eta} - \frac{\partial f}{\partial \eta} \frac{\partial g}{\partial \xi} \right).$$

We recover the full system by way of the conjugate variables ξ_0 and P,

$$\dot{\xi}_0 = \frac{\partial H}{\partial P}$$

Parameters

- P^* — Impulse of system
- B_1^* — Thickness of rings relative to L
- β — Relative thickness of rings
- γ — Relative strength of rings

Recap the canonical reduction

$$H(Z_1, R_1, Z_2, R_2; \Gamma_1, \Gamma_2, B_1, B_2, L)$$

Equations of motion:

$$\dot{Z}_1 = \{Z_1, H\} = \frac{1}{\Gamma_1 R_1}\frac{\partial H}{\partial R_1}$$
$$\dot{R}_1 = \{R_1, H\} = -\frac{1}{\Gamma_1 R_1}\frac{\partial H}{\partial R_1}$$
$$\dot{Z}_2 = \{Z_2, H\} = \frac{1}{\Gamma_2 R_2}\frac{\partial H}{\partial R_2}$$
$$\dot{R}_2 = \{R_2, H\} = -\frac{1}{\Gamma_2 R_2}\frac{\partial H}{\partial R_2}$$

with the modified Poisson bracket

$$\{f, g\} = \sum_{j=1}^{2} \frac{1}{\Gamma_j R_j}\left(\frac{\partial f}{\partial Z_j}\frac{\partial g}{\partial R_j} - \frac{\partial f}{\partial R_j}\frac{\partial g}{\partial Z_j}\right)$$

$$H(\xi, \eta; P^*, B_1^*, \beta, \gamma)$$

Equations of motion:

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$$H(\xi, \eta; P^*, B_1^*, \beta, \gamma)$$

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We recover the full system by way of the conjugate variables ξ_0 and P,

$$\dot{\xi}_0 = \frac{\partial H}{\partial P}$$

Integrability of the N-vortex problem

N	Point vortices on the plane	Coaxial vortex rings	Coaxial vortex ring arrays
1	stationary	cyclic/trivial	cyclic/trivial
2	cyclic/trivial	integrable	integrable
3	integrable	chaotic	chaotic
4	chaotic	chaotic	chaotic

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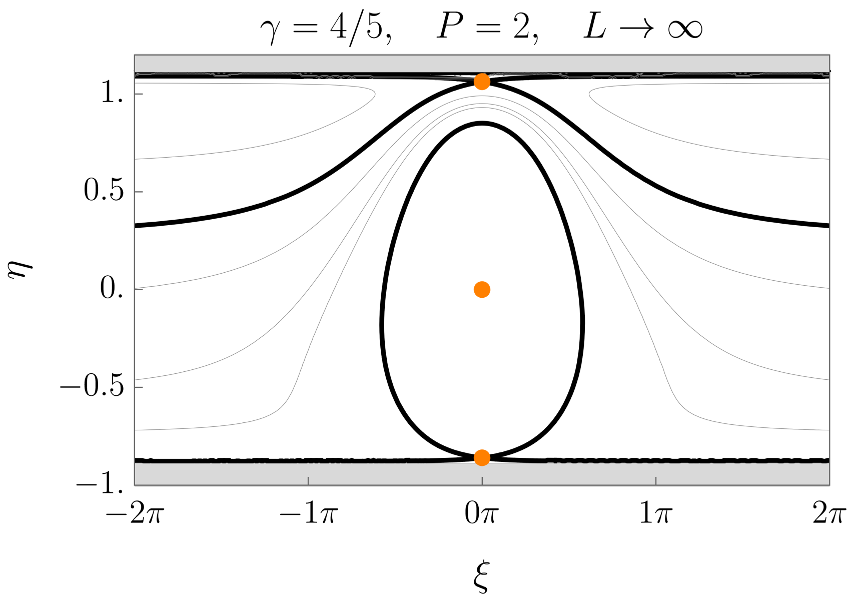
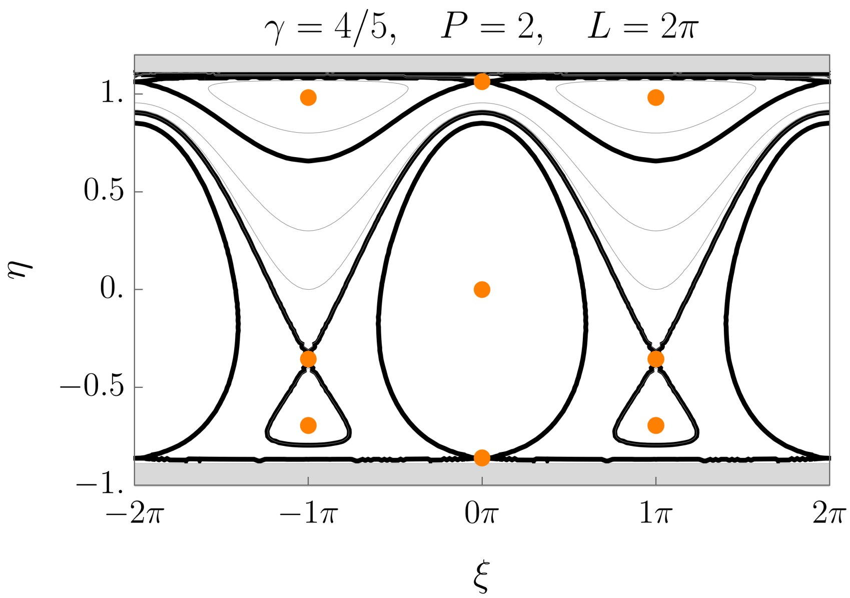
The reduced Hamiltonian for two coaxial vortex rings

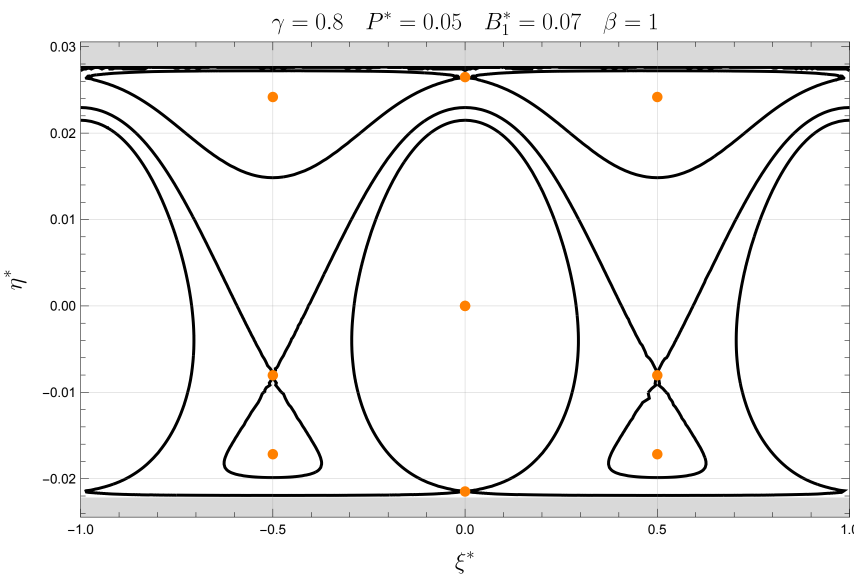


reproduces a figure from Borisov, Kilin, and Mamaev 2013

The reduced Hamiltonian for two coaxial vortex ring arrays



Phase portraits in non-dimensional coordinates ξ^* and η^*



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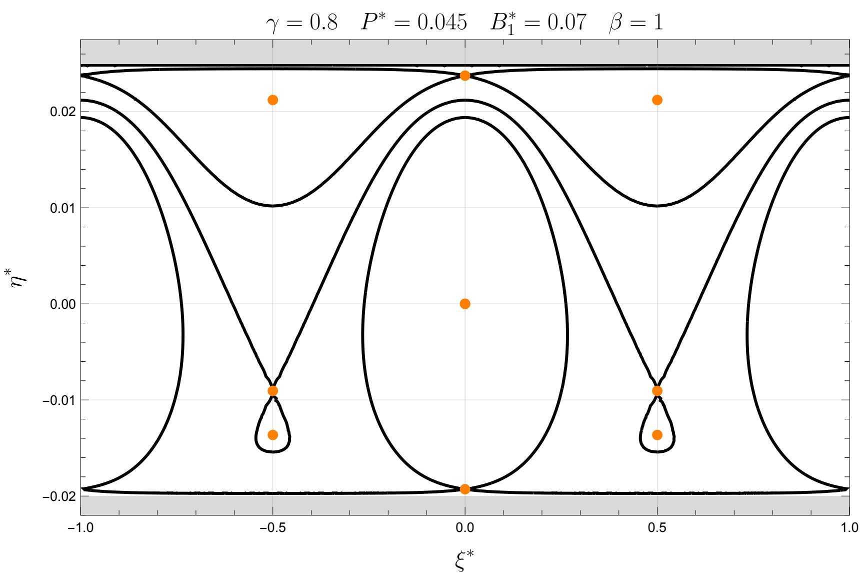

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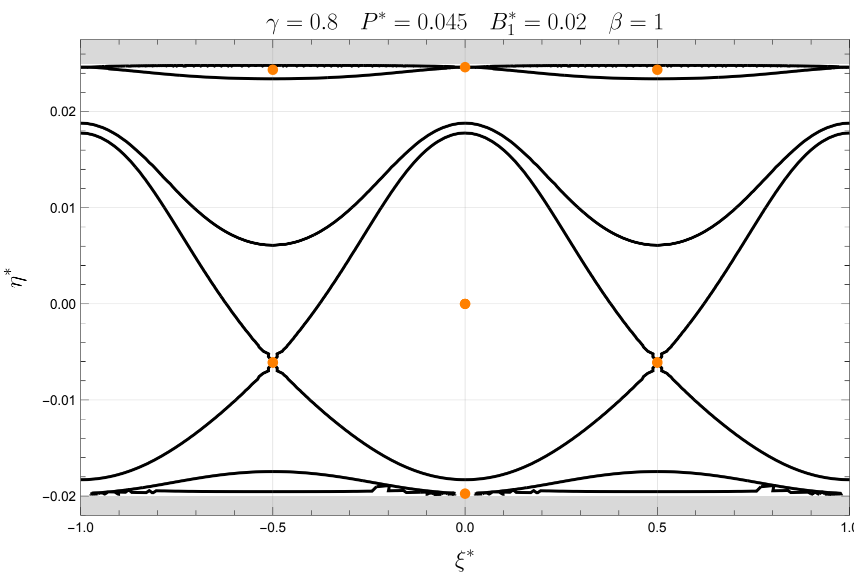
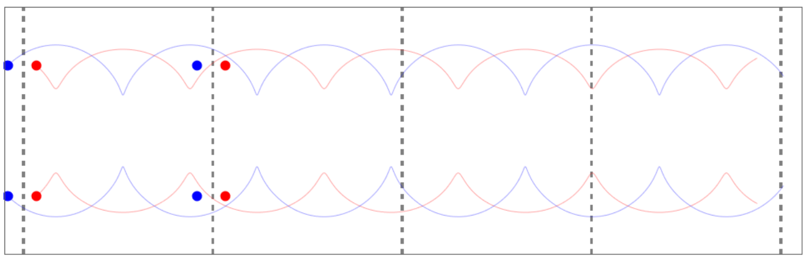
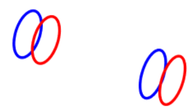
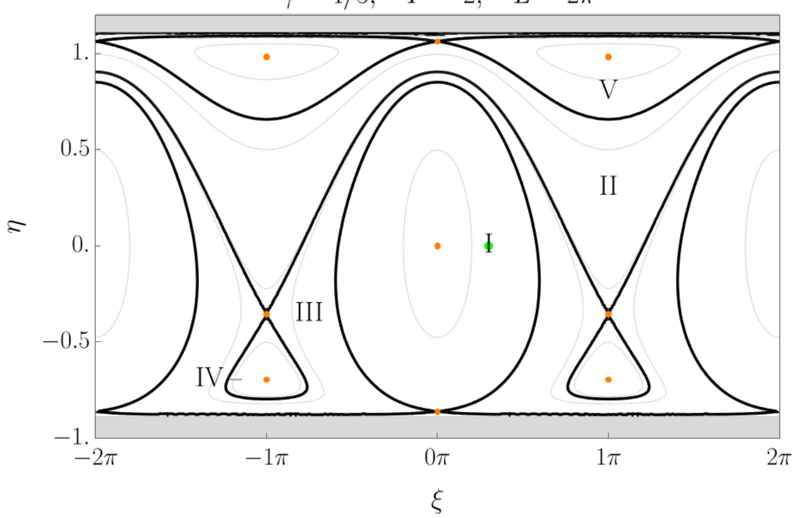


Regime I: Leapfrogging

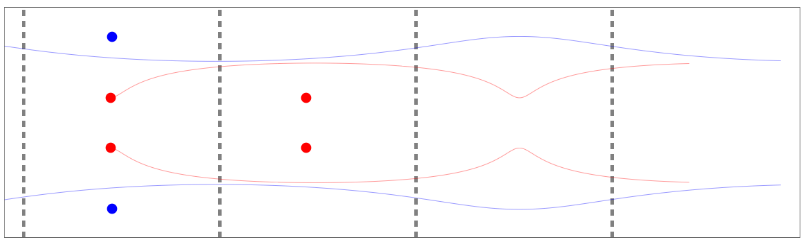


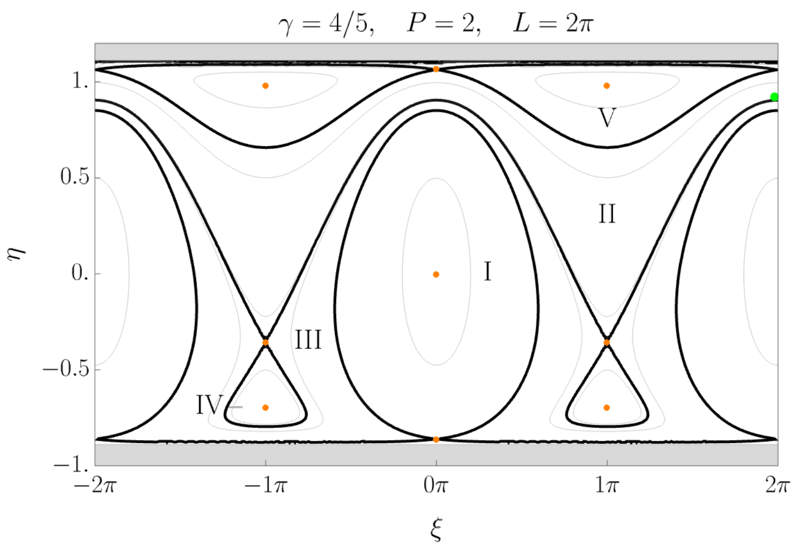


$\gamma = 4/5, \quad P = 2, \quad L = 2\pi$

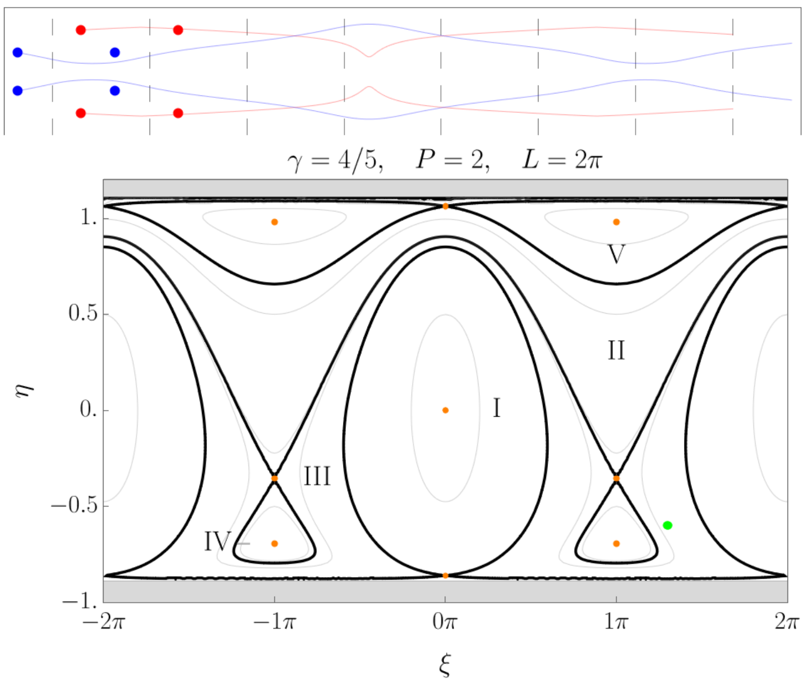


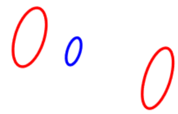
Regime II: Secular Pass-Through



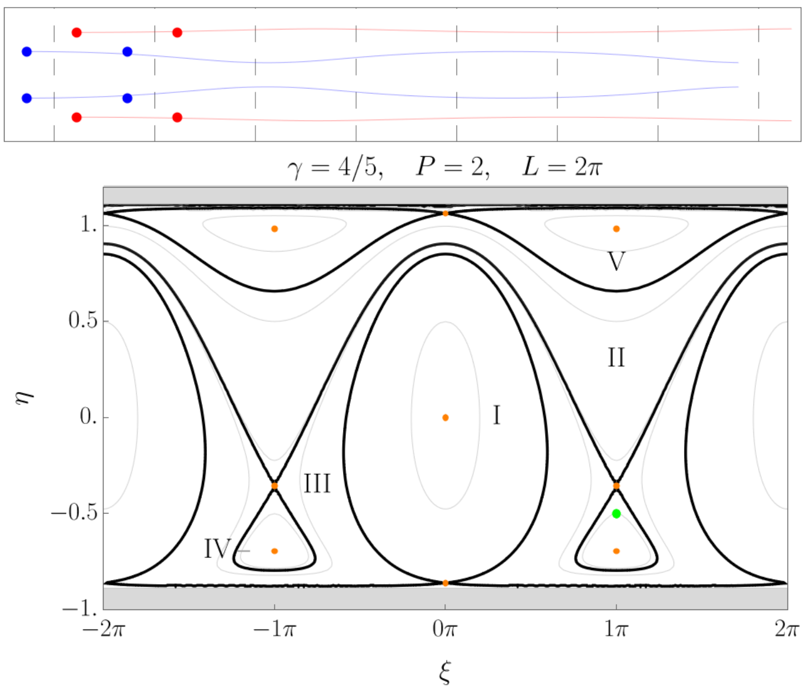


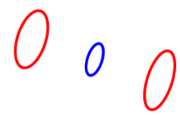
Regime III: Retrograde Pass-Through



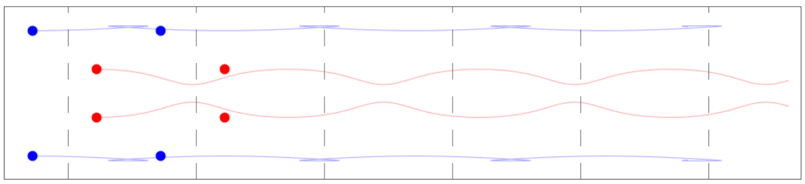


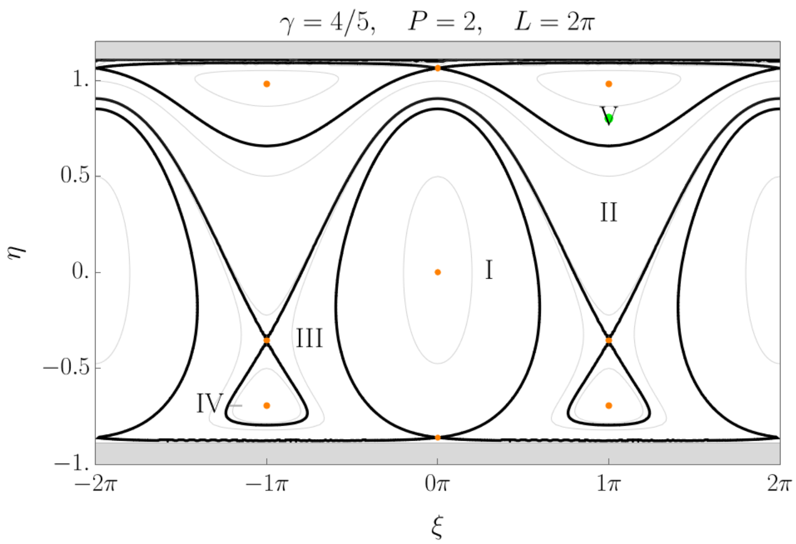
Regime IV


$\gamma = 4/5, \quad P = 2, \quad L = 2\pi$
I
II
III
IV
V
1.
0.5
0.
−0.5
−1.
η
−2π
−1π
0π
1π
2π
ξ

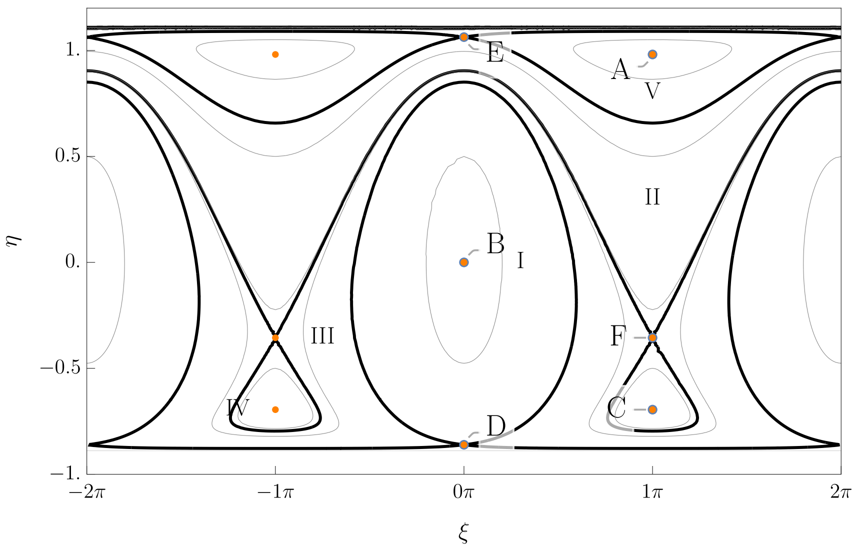


Regime V

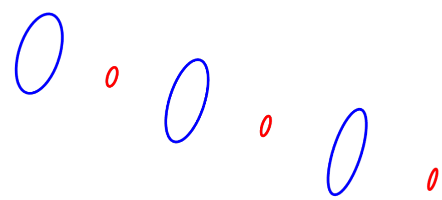




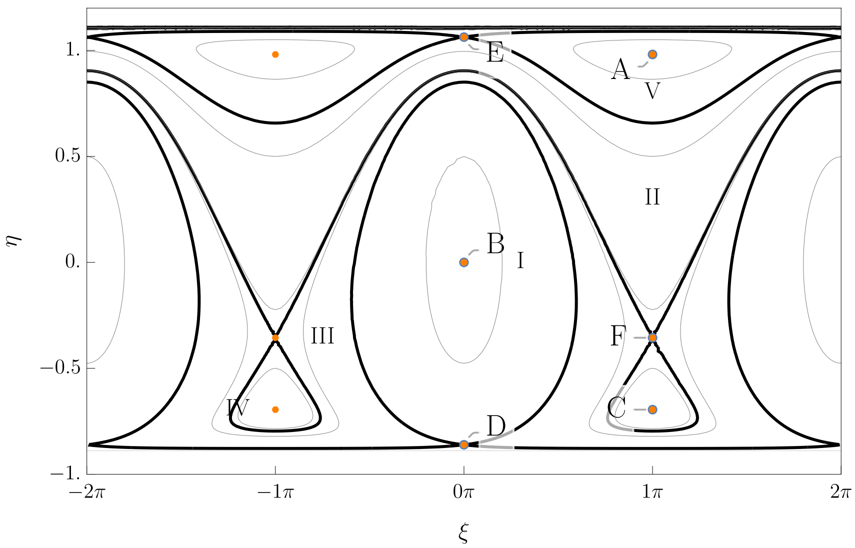
Relative Equilibria



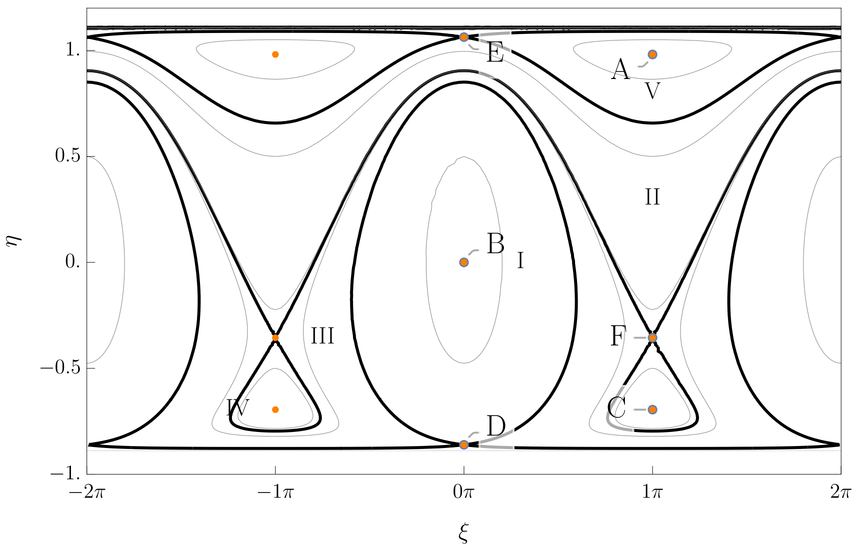
Equilibrium A



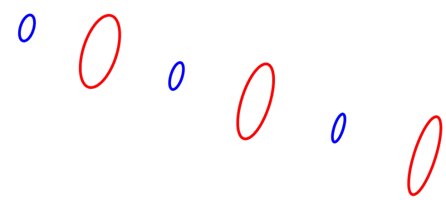
Relative Equilibria



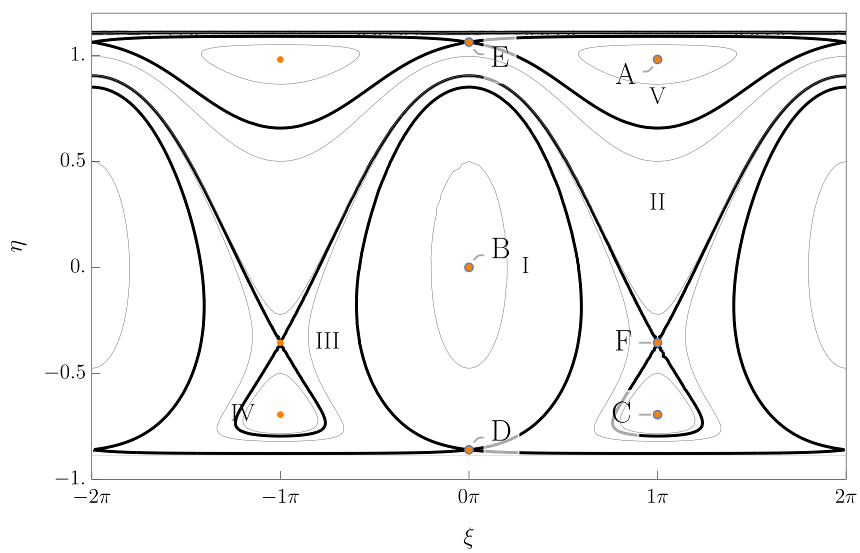
Equilibrium B



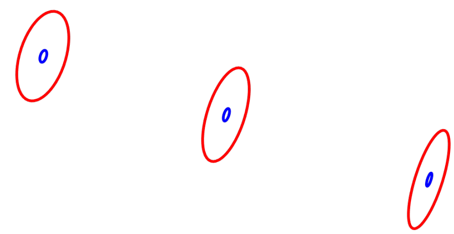
Equilibrium C



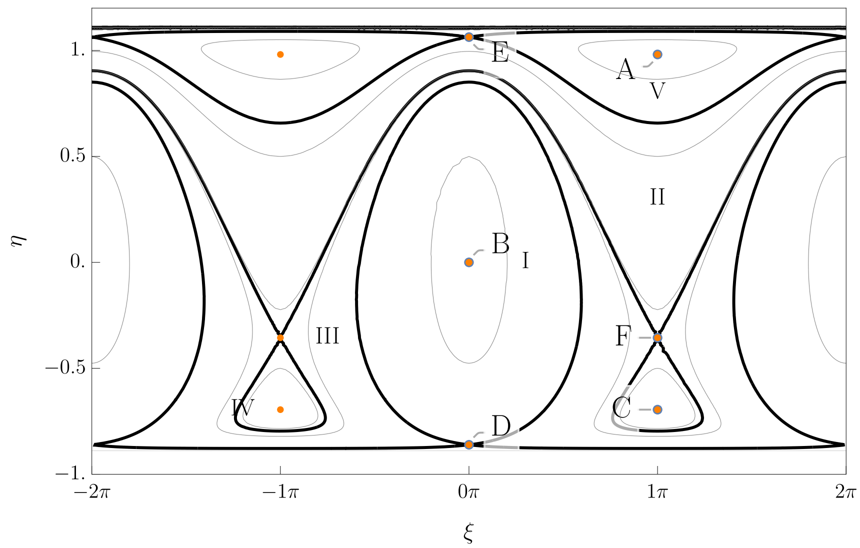
Relative Equilibria



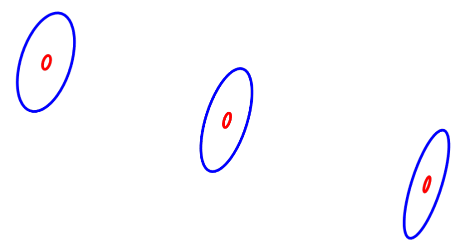
Equilibrium D

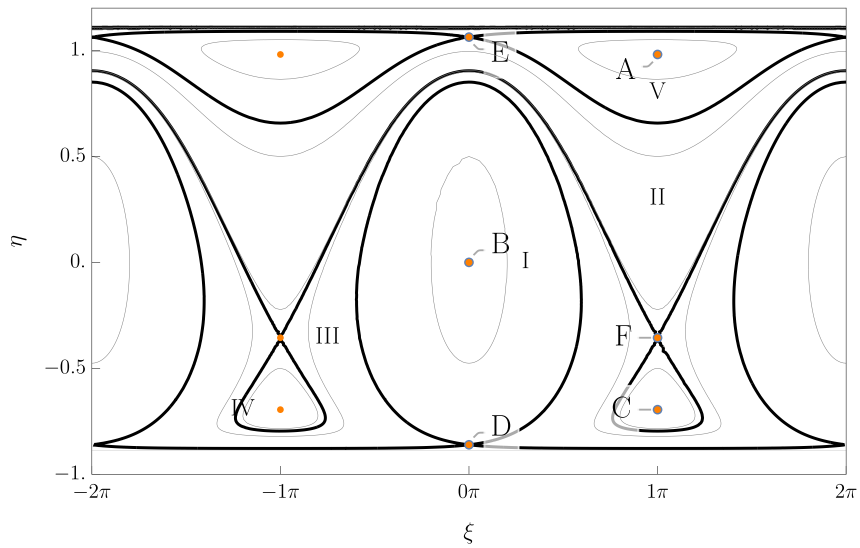


Relative Equilibria

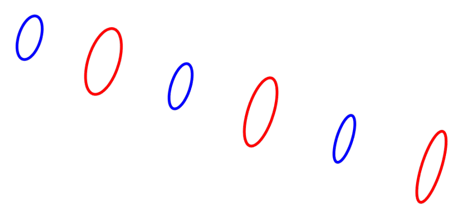


Equilibrium E





Equilibrium F



- Arrays of coaxial vortex rings — the periodic N-vortex ring problem
- Streamfunction in a co-moving reference frame for a single array
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