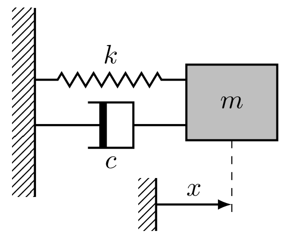
1. Governing equations for a simple harmonic oscillator. The simple harmonic oscillator is governed by the following second-order ordinary differential equation.

$$m\ddot{x} + c\dot{x} + kx = 0, \tag{1a}$$

where the symbols can be interpreted to have the meaning shown in the diagram below.



We can supplement the differential equation (1a) with two initial conditions, on the position and velocity respectively,

$$x(0) = x_0, \tag{1b}$$
$$\dot{x}(0) = v_0. \tag{1c}$$

to obtain a complete initial value problem (1).

(a) Identify the 'state variables' of the dynamical system given by (1a).
There are two state variables. The 'state variables' are the variables whose values define the 'state' of the system at any time t; they can be thought of as the things whose rates of change are given by the differential equation(s) under consideration.
If you have trouble thinking through this part, move on to the next one, and come back to this later.

Ans. The state variables are x and $\dot{x}$.

(b) Write down a first-order dynamical system with $n = 2$ that is equivalent to eq. (1a). Do this in three different ways.

i. Write two separate scalar differential equations.
ii. Write a vector differential equation, i.e., write an equation where the left hand side is a column vector, and the right hand side is also a column vector.
iii. Write a matrix-vector differential equation, i.e., write an equation where the left hand side is a column vector, and the right hand side is a matrix (that does **not** contain the state variables) multiplied by a column vector (whose elements are the state variables).

Ans. Let the two state variables be named $y_1 (= x)$ and $y_2 (= \dot{x})$. This is not necessary, but it often helps to clarify the process. Since the rate of change of x is $\dot{x}$ by definition, we can write

$$\dot{y_1} = y_2. \tag{2a}$$

It sounds like this shouldn't be a very meaningful equation, but it actually is useful. Next, we translate (1a) in terms of y_1 and y_2, i.e., $m\dot{y_2} + cy_2 + ky_1 = 0$, which can be re-written as

$$\dot{y_2} = -\frac{c}{m}y_2 - \frac{k}{m}y_1. \tag{2b}$$

i. Equations (2) are the two separate scalar differential equations.
ii. We can re-write (2) as follows:

$$\begin{bmatrix} \dot{y_1} \\ \dot{y_2} \end{bmatrix} = \begin{bmatrix} y_2 \\ -(c/m)y_2 - (k/m)y_1 \end{bmatrix} \tag{3}$$

iii. This time, we have to extract a matrix from the right hand side of (4). By inspection, we can arrive at the following equation.

$$\begin{bmatrix} \dot{y_1} \\ \dot{y_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{4}$$

2. Analytical solution of the governing equations for a simple harmonic oscillator. In this problem, you will use your knowledge of differential equations to solve the differential equation (1a). For this problem, you should ignore the initial conditions (1b) and (1c). Instead, use the ansatz

$$x(t) = \exp \lambda t, \tag{5}$$

where λ has both a real and an imaginary part. All physical quantities are to be interpreted as the real part of corresponding complex numbers. Thus, both $x(t)$ and $\dot{x}(t)$ will in general be complex, and we interpret them physically using their real part.
Analytically solve the equation (1a) for the position $x(t)$ and velocity $\dot{x}(t)$ using the ansatz (5). Give your answer in terms of k, m, c and t (not λ). There may be more than one correct expression for the position and the velocity.

Ans. From (5), it follows that $\dot{x} = \lambda \exp \lambda t$ and $\ddot{x} = \lambda^2 \exp \lambda t$. Substituting these into (1a), we can write

$$\begin{aligned} m\lambda^2 e^{\lambda t} + c\lambda e^{\lambda t} + k e^{\lambda t} &= 0 \\ \implies m\lambda^2 + c\lambda + k &= 0. \end{aligned}$$

Since this is a quadratic equation for λ, we can use the quadratic formula (this is why you did math in high school!)

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.$$

For λ to have both a real and imaginary part, $c^2 < 4mk$. Assuming that this inequality holds, we can write λ out in complex form as

$$\lambda = -\frac{c}{2m} \pm i\frac{\sqrt{4mk - c^2}}{2m},$$

where this time the quantity under the square root is positive. Now, we are ready to substitute λ into (5) and to then take its real part.

$$x(t) = e^{-\dfrac{c}{2m}t \pm i\dfrac{\sqrt{4mk-c^2}}{2m}t}.$$

Using convenient symbols a and b to refer to the real and imaginary parts of λ respectively, we can re-write the above as

$$x(t) = e^{at \pm ibt},$$

and using the fact that $e^{i\theta} = \cos\theta + i\sin\theta$,

$$\begin{aligned} x(t) &= e^{at} \cdot e^{\pm ibt} \\ &= e^{at} \left[\pm \cos bt \pm i \sin bt \right], \end{aligned}$$

and taking the real part of x, we find that the position as a function of time is

$$\begin{aligned} & e^{at} \left[\pm \cos bt \right] \\ &= \left[e^{-\left(\dfrac{c}{2m}\right)t} \right] \cdot \left[\pm \cos\left(\frac{\sqrt{4mk - c^2}}{2m} t \right) \right]. \end{aligned} \tag{6}$$

To do the same for velocity $\dot{x}(t)$, we could either choose to start from the complex exponential expression, $\dot{x}(t) = \lambda e^{\lambda t}$, or we could differentiate the above expression. You should be comfortable with using either approach.

The first approach using the complex exponential expression $\dot{x}(t) = \lambda e^{\lambda t}$.

$$\begin{aligned}
\dot{x}(t) = \lambda e^{\lambda t} &= \left[-\frac{c}{2m} + i\frac{\sqrt{4mk-c^2}}{2m}\right] \cdot \left[e^{-\frac{c}{2m}t + i\frac{\sqrt{4mk-c^2}}{2m}t}\right] \\
\text{or } &= \left[-\frac{c}{2m} - i\frac{\sqrt{4mk-c^2}}{2m}\right] \cdot \left[e^{-\frac{c}{2m}t - i\frac{\sqrt{4mk-c^2}}{2m}t}\right] \\
&= [a+ib] \cdot \left[e^{at+ibt}\right] = [a+ib] \cdot \left[e^{at} \cdot e^{ibt}\right] = [a+ib] \cdot e^{at} \cdot [\cos bt + i \sin bt] \\
\text{or } &= [a-ib] \cdot \left[e^{at-ibt}\right] = [a-ib] \cdot \left[e^{at} \cdot e^{-ibt}\right] = [a-ib] \cdot e^{at} \cdot [-\cos bt - i \sin bt] \,.
\end{aligned}$$

Taking the real part of the above, we find that the velocity can be written as

$$\begin{aligned}
e^{at} \cdot \left[\ \ a\cos bt + i^2 b \sin bt\right] &= e^{at} \cdot [\ \ a\cos bt - b\sin bt] \\
\text{or } e^{at} \cdot \left[-a\cos bt + i^2 b \sin bt\right] &= e^{at} \cdot [-a\cos bt - b\sin bt]
\end{aligned}$$

which can be summarized as

$$e^{at}\left[-b\sin bt \pm a\cos bt\right].$$

Once again replacing a and b with the real and imaginary parts of λ respectively, we find an expression for the velocity being the real part of $\dot{x}$,

$$\left[e^{-\left(\frac{c}{2m}\right)t}\right] \cdot \left[-\frac{\sqrt{4mk-c^2}}{2m}\sin\left(\frac{\sqrt{4mk-c^2}}{2m}t\right) \pm \left(-\frac{c}{2m}\right)\cos\left(\frac{\sqrt{4mk-c^2}}{2m}t\right)\right] \tag{7}$$

Whew, that was a lot. Let's see if the other approach is somewhat less cumbersome.

The second approach differentiating (6). This way, we won't have any complex numbers to deal with. Using the product rule, we find that the derivative of $e^{at}\cos bt$ is

$$ae^{at}\cos bt - e^{at} b \sin bt,$$

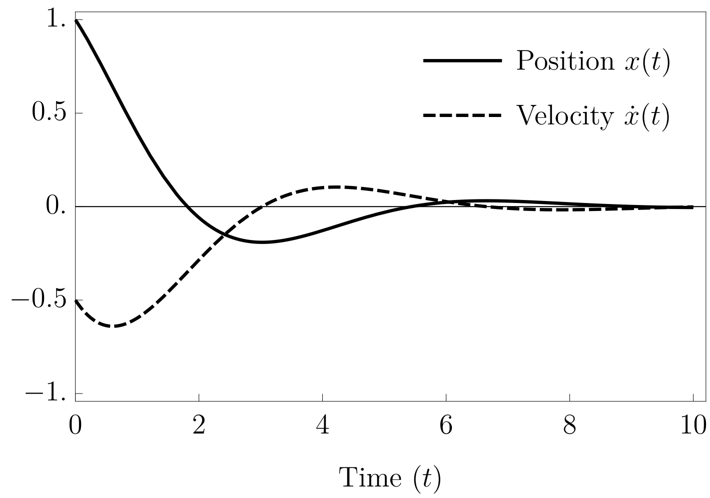
and it is possible to see that this corresponds to the same as the expression above (7).

3. Plotting the analytical solution. For the case when $m = k = c = 1$, show that one of the solutions to (1a) together with (5) can be graphically represented as shown here. To answer this question successfully, you should reproduce the plot below using your answer to problem 2. Note that this problem does **not** ask you to find a **numerical solution** to (1a). Instead, you are asked to use a computer to plot the analytical solution to (1a) that was found in problem 2.

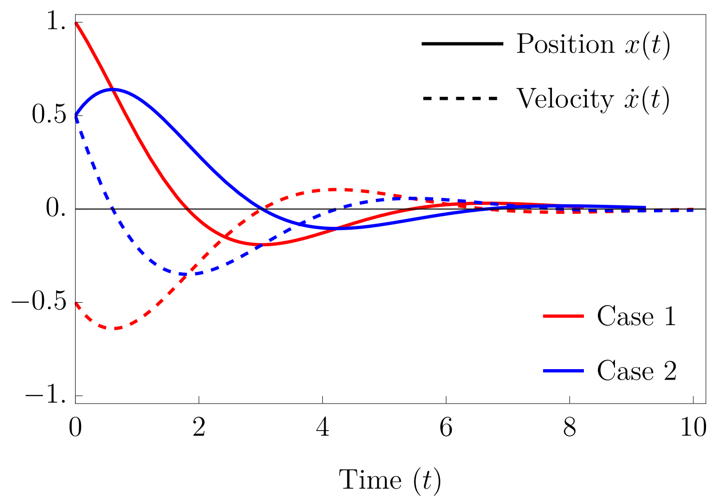
Ans. This question is ultimately about plotting the expressions (6) (7). Let's make our lives easier by substituting $m = k = c = 1$ so that the coefficients become somewhat simpler. With this assumption, the position (6) and the velocity (7) become, respectively,

$$\begin{aligned}
&\text{Position} \quad \pm\, e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) \\
&\text{Velocity} \quad e^{-t/2}\left(-\frac{1}{2}\sqrt{3}\sin\left(\frac{\sqrt{3}t}{2}\right) \mp \frac{1}{2}\cos\left(\frac{\sqrt{3}t}{2}\right)\right).
\end{aligned}$$

Choosing the positive sign from '±' in the expression for position and the negative sign from '∓' in the expression for velocity gives the two graphs shown below. Other choices are also valid.



4. Numerical solution of ODEs (see tutorial at `https://emadmasroor.github.io/classes/E91_S25/Resources/NumericalSolution`). For the case when $m = k = c = 1$, numerically solve the differential equation (1a) using a program of your choice, and plot the resulting solutions — both $x(t)$ and $\dot{x}(t)$ — for $t \in [0, 10]$. Do this for two cases:
   - Case 1: $x(0) = 1, \quad \dot{x}(0) = -1/2$, and
   - Case 2: $x(0) = \dot{x}(0) = 1/2$.



You should be able to reproduce the figure above. Turn in the code you use to solve the problem as an additional file.

Ans. Mathematica code is given below to reproduce the figure above.

```
In[1]:= sol1 = NDSolve[{x''[t] + x'[t] + x[t] == 0, x[0] == 1, x'[0] == -1/2},
        {x[t], x'[t]}, {t, 0, 10}]

In[2]:= sol2 = NDSolve[{x''[t] + x'[t] + x[t] == 0, x[0] == 0.5, x'[0] == 1/2},
        {x[t], x'[t]}, {t, 0, 10}]

In[3]:= Plot[{{x[t] /. sol1, x'[t] /. sol1, x[t] /. sol2, x'[t] /. sol2}}, {t, 0, 10}]
```