1. A geometric way of thinking about dynamics. For each of the following dynamical systems of dimension $n = 1$,

   (a) Draw a phase portrait similar to the one drawn in class for $\dot{x} = \sin x$.

   (b) Identify the fixed points, and decide whether each is an 'attractor' or a 'repeller'.

   (c) List the qualitatively different things that can happen to x over time depending on where on $\mathbb{R}$ the flow is initialized (i.e., depending on the initial condition).

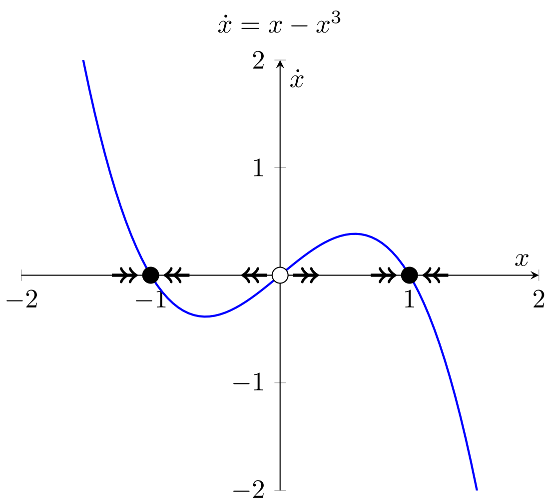
   Answer (a) through (c) for:

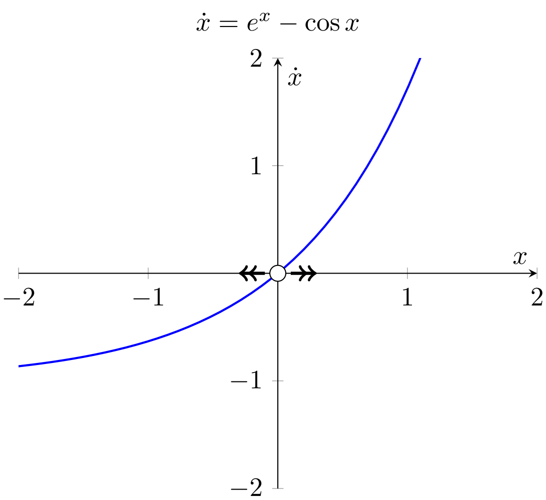
$$\dot{x} = x - x^3 \tag{1}$$
$$\dot{x} = e^x - \cos x \tag{2}$$

   Your answers should be in the form of a neatly drawn sketch or plot, and complete sentences. For example, you could write

   > "System (2) has two fixed points: an attractor at A and a repeller at B as shown in the diagram. If x is initialized between 0 and 3, it will increase until it reaches B."

Ans. (a) The phase portraits are shown below.





(b) For eq. (1), there is one repelling fixed point at $x = 0$ and two attracting fixed points at $x = \pm 1$. For eq. (2), there is a single repelling fixed point at $x = 0$.

(c) In system (1), if x is initialized with a value less than 0, it ends up at $x \approx -1$; if it is initialized with a value greater than 0, it ends up at $x \approx +1$. In system (2), $x \to +\infty$ if x is initalized with a value greater than 0, and $x \to -\infty$ is x is initalized with a value less than 0.

2. For the logistic equation

$$\dot{N} = rN\left(1 - \frac{N}{K}\right), \tag{3}$$

(a) determine the value of N at which N increases the fastest, in terms of parameters r and K.

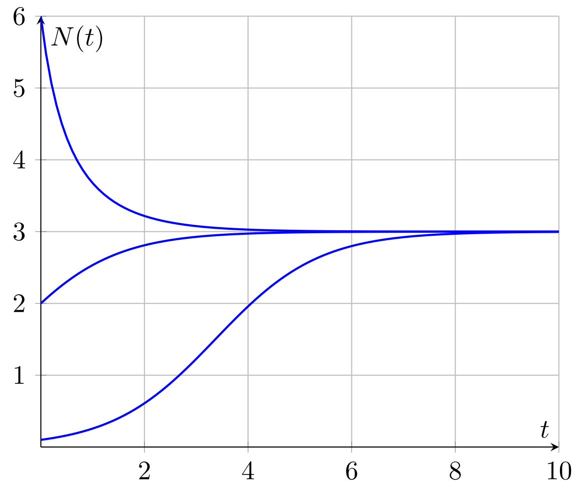
(b) Numerically solve (3) for a reasonable value of r and of K with three different initial conditions of your choosing. A computer-generated plot is sufficient to answer this part. Turn in your code also.

Ans. For N to increase the fastest, the derivative of $\dot{N}$ with respect to N should be equal to zero. This gives

$$-\frac{rN}{K} + r\left(1 - \frac{N}{K}\right) = 0$$
$$\implies N = \frac{K}{2}$$

A numerical solution to this differential equation, with K, the carrying capacity, set to three units of N and r, the growth rate, set to 1 unit of inverse time, is shown below with three initial conditions.

Evolution of solutions to (3) with $r = 1$, $K = 3$



3. The growth of certain tumors can be modeled using the equation

$$\dot{N} = -aN\log(bN), \tag{4}$$

where log stands for the natural logarithm. $N(t)$ is proportional to the number of cells in the tumor.

(a) What could the parameters a and b represent in real-world terms?

(b) For $a = 1.2$ and $b = 1.0$, draw a phase portrait and sketch the graph of $N(t)$ for three initial values: $N(0) = \{0.1, 0.6, 1.6\}$. You should produce these plots with a computer program of your choice.

(c) Explain (in complete sentences) what happens to the number of tumor cells as time passes for each of the initial values from 3b.

Ans. (a) It seems that the constant a is proportional to the growth rate of the tumor cells, whereas b is inversely proportional to the size of the organ in which the tumor has developed, or the 'carrying capacity' of the tumor cells.

(b) Phase portrait and trajectories.

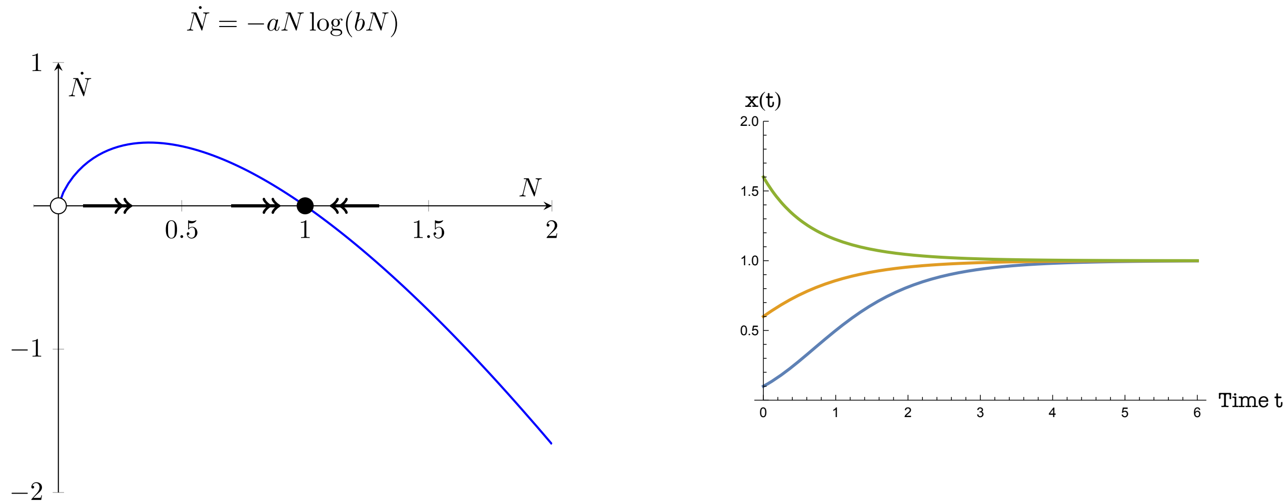


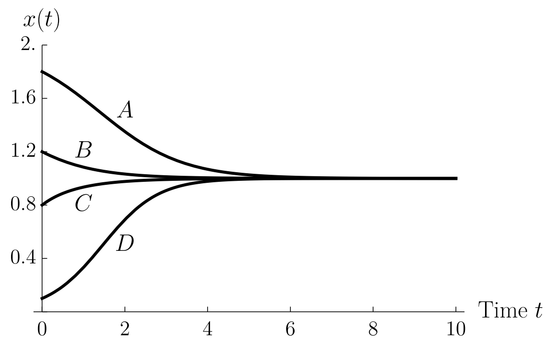
Figure 1: Phase portrait and trajectories for eq. (4) with $a = 1.2, b = 1$.

(c) If the cancer cells start out with $N = 0.1$, they rapidly increase in number — first at an increasing rate until about $t = 1$, then at a decreasing rate until by about $t \approx 5$, the number settles at 1.0. If the number of cells starts out with $N = 0.6$, the number of cells increases with a slowing rate and eventually settles at $N = 1.0$ at about $t \approx 5$ units. Finally, if the number of cells starts out at $N = 1.6$, the number decreases at a slowing rate of decrease, eventually settling at $N = 1.0$ at about $t \approx 5$ units.

4. Consider the dynamical system

$$\dot{x} = (x-1)(x-2). \tag{5}$$

Which of the following, if any, are possible trajectories $x(t)$ arising from this dynamical system? For those that are possible, indicate on a phase portrait of system (5) where they are initialized. For those that are not possible, explain how you arrived at this conclusion.



Ans. The phase portrait of the system (5) is shown in fig. **??**. According to this phase portrait, any initial condition that starts below $x = 2$ will eventually end up at the fixed point $x = 1$. Thus, all four trajectories A through D seem like they are possible.

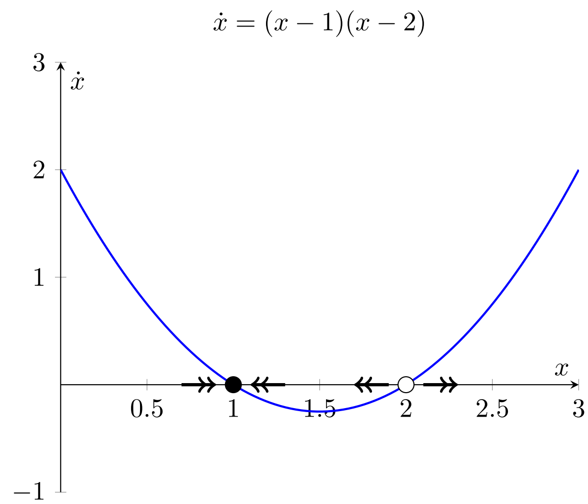


Figure 2: Phase portrait for (5)