1. For the system

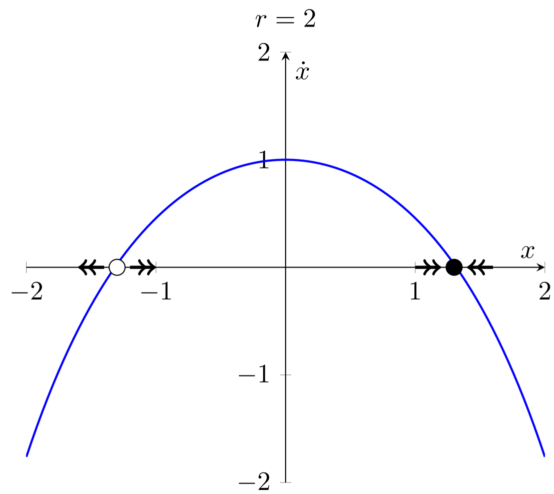
$$\dot{x} = f(x; r) = r - \cosh x, \tag{1}$$

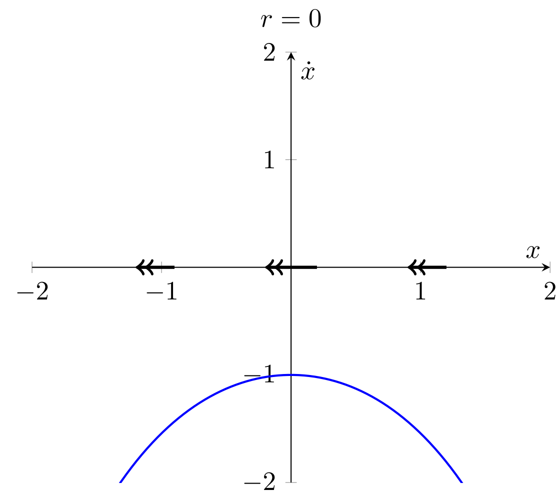
   (a) Draw all the qualitatively different phase portraits on separate axes for different values of r. In each phase portrait, draw arrows on the x axis to show the direction of 'flow', and mark the fixed points. Note the stability or instability of each fixed point. **Note:** you should only need one, two or three different values of r to cover all qualitatively different cases. Ignore any phase portraits with semi-stable fixed points; these are 'edge cases' that are not very important for now.

   (b) Draw a bifurcation diagram, i.e., a plot with r on the horizontal axis and x on the vertical axis, with a curve (or curves) showing the fixed points x^* that exist for each value of r. Recall that fixed points are those values of x for which $f(x) = 0$. Use dashed lines for the unstable fixed points and solid lines for the stable fixed points.

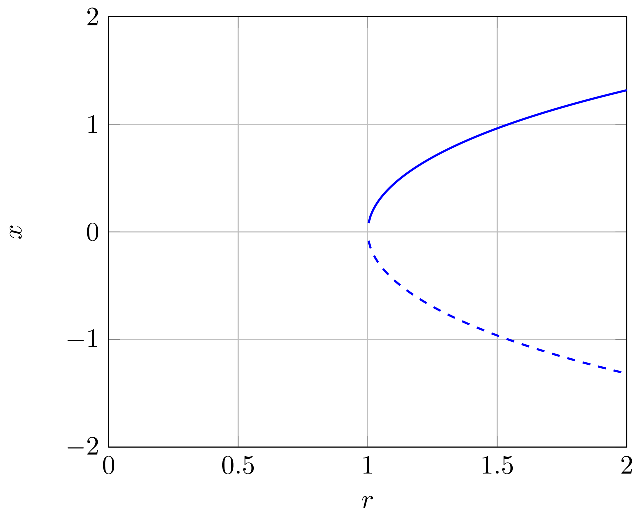
   (c) Determine the critical value of r at which a bifurcation occurs.

Ans. (a) The two different kinds of phase portraits are shown below. Unstable fixed points are shown with ○ and stable fixed points are shown with ●.



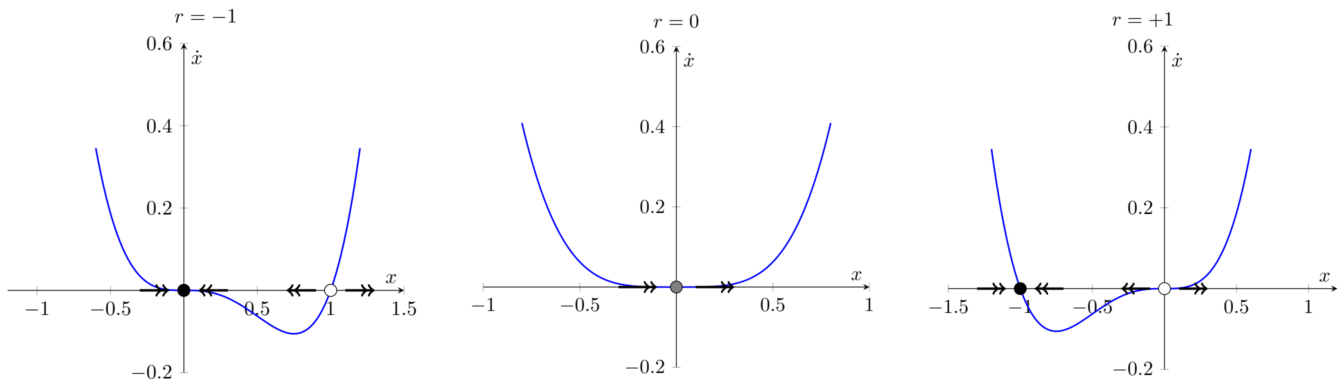


(b) The bifurcation diagram is drawn below.

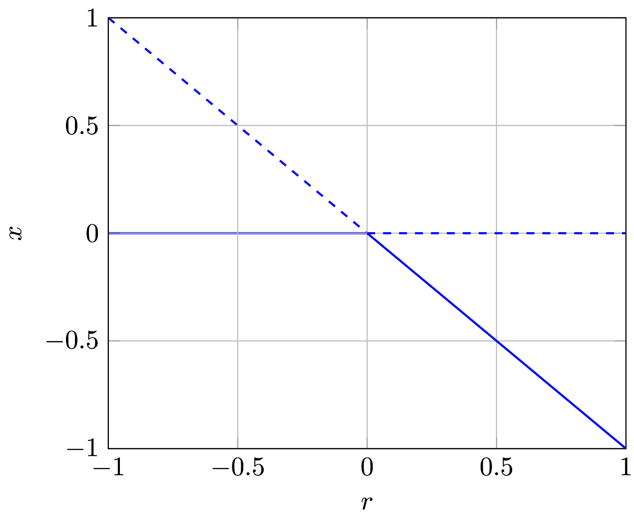


2. For the system
$$\dot{x} = f(x; r) = rx^3 + x^4, \tag{2}$$
   (a) Draw all the qualitatively different phase portraits on separate axes for different values of r. In each phase portrait, draw arrows on the x axis to show the direction of 'flow', and mark the fixed points. Note the stability or instability of each fixed point. **Note:** This time, you should show semi-stable fixed points as well as stable and unstable ones.
   (b) Sketch a bifurcation diagram, i.e., a plot with r on the horizontal axis and x on the vertical axis, with a curve (or curves) showing the fixed points x^* that exist for each value of r. Recall that fixed points are those values of x for which $f(x) = 0$. Use dashed lines for the unstable fixed points and solid lines for the stable fixed points.
   (c) Determine the critical value of r at which a bifurcation occurs.

Ans. (a) The phase portraits are drawn below. Unstable fixed points are shown with ○ and stable fixed points are shown with ●. Semistable fixed points are shown with ●.



(b) The bifurcation diagram is shown below.



3. Consider two systems governed by the differential equations

$$\dot{x} = f(x; r) = rx \pm x^3, \tag{3}$$

where one of the systems has a '+' sign and the other has a '−' sign. These systems undergo pitchfork bifurcations as the parameter r is varied from below $r = 0$ to above $r = 0$. The bifurcation diagrams for supercritical and subcritical pitchfork bifurcations are shown in fig. 1 for your reference. If both systems are

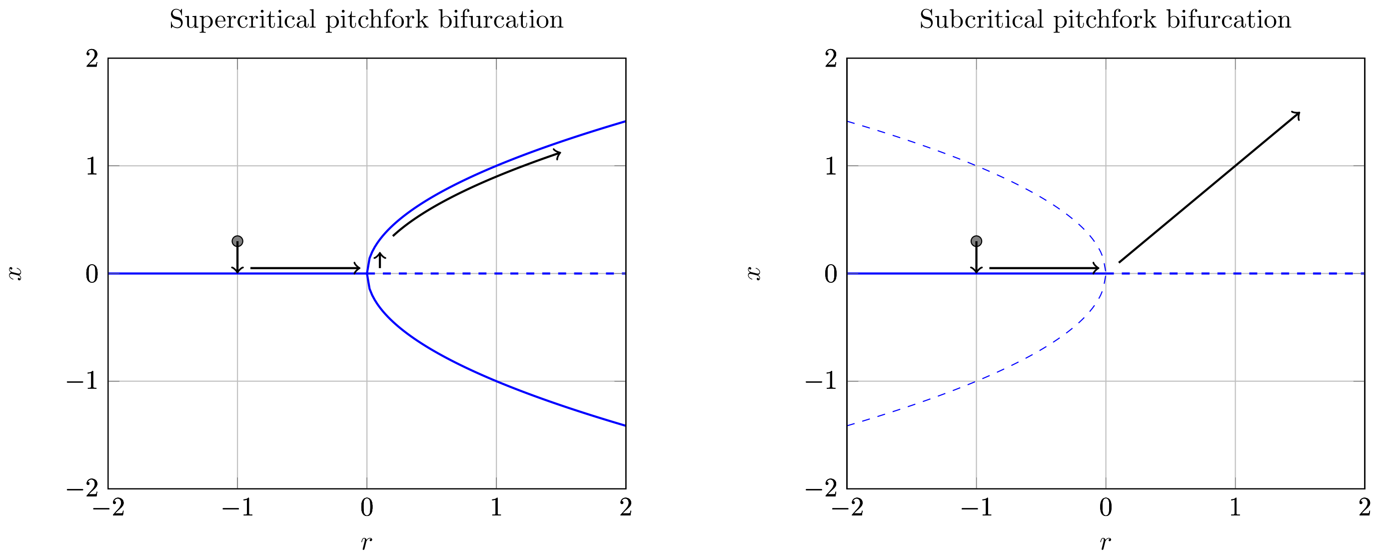


Figure 1: Bifurcation diagrams for the pitchfork bifurcation

initialized with $x(0) = 0.3$ — indicated with ● — use the bifurcation diagrams to explain, with words and sketches as necessary, what will happen to $x(t)$ as time passes and r is slowly increased from $r = -1$ to $r = +1$. You must write two different explanations, one for the system with a supercritical pitchfork bifurcation and one for the system with a subcritical pitchfork bifurcation. You should not need more than a few sentences for each system. In your answer, you should allow for the possibility that there will be small disturbances to the value of $x(t)$, as is almost always true for systems in engineering.

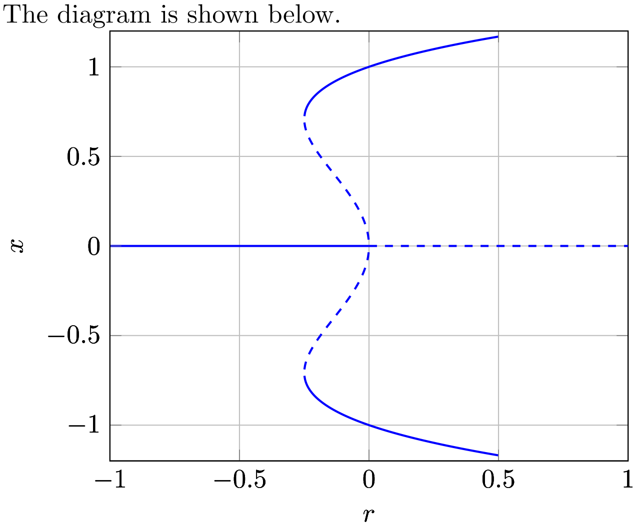
Ans. Supercritical pitchfork bifurcation. Initially, $x(t)$ will rapidly decrease until it reaches $x = 0$, and will remain close to $x = 0$ since $x = 0$ is a stable fixed point. As r is increased from -1 to $0-$, not much of a change will be observed, and x will remain close to $x = 0$. Once $r > 0$, $x = 0$ will become unstable, and therefore x will rapidly increase until it reaches the stable fixed point at $x = +\sqrt{r}$. Subsequently, x will remain close to this fixed point; as r increases further, x will slowly increase in proportion to $\sqrt{r}$.

Subcritical pitchfork bifurcation. Initially, $x(t)$ will rapidly decrease until it reaches $x = 0$, and will remain close to $x = 0$ since $x = 0$ is a stable fixed point. We might expect the decrease of x in this stage to be faster than in the supercritical bifurcation, because it is both 'pulled' by the stable fixed point at $x = 0$ and 'pushed' by the unstable fixed point at $x = \sqrt{-r}$. As r is increased from -1 to $0-$, not much of a change will be observed, and x will remain close to $x = 0$. Once $r > 0$, $x = 0$ will become unstable, and therefore x will rapidly increase. This time, there is no stable fixed point to 'catch' x, and it will continue to increase without bound once r is even slightly larger than 0. Further increases of r will cause no qualitative changes.

A sketch of each process is shown in fig. 1.

4. Draw the bifurcation diagram for
$$\dot{x} = rx + x^3 - x^5, \tag{4}$$
with r in the range $-0.5 < r < 0.5$. Your diagram should have dashed lines for unstable fixed points, solid lines for stable fixed points, and should be qualitatively correct with respect to the direction and curvature of the lines that you draw.

Ans. The diagram is shown below.



The curves were produced by solving the equation $rx + x^3 - x^5 = 0$ for x in terms of r, which yields
$$x^* = \frac{\pm\sqrt{1 \pm \sqrt{1+4r}}}{\sqrt{2}}. \tag{5}$$