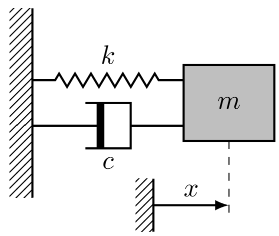
1. The mass-spring-dashpot system, shown below, can be modeled using the linear system that you investigated in HW 1. The governing equations can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}, \tag{1}$$

where x is the position measured relative to the rest length of the spring and v is the velocity; both quantities have signs that indicate the direction (right or left). This system can be written succinctly as

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}. \tag{2}$$



(a) Calculate the eigenvalues of the system matrix A in terms of the ratios k/m and c/m. (We have combined the three quantities m, k and c into the two ratios k/m and c/m.)

Ans. The eigenvalues can be calculated using the characteristic equation $\det(A - \lambda \boldsymbol{I}) = 0$. Thus, we get

$$\left| \begin{bmatrix} 0 - \lambda & 1 \\ -k/m & -c/m - \lambda \end{bmatrix} \right| = 0 \implies \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0,$$

and let us label the two ratios as c_m and k_m respectively for conciseness. We then get the solution

$$\lambda = \frac{-c_m \pm \sqrt{c_m^2 - 4k_m}}{2}.$$

(b) Also calculate τ and Δ, defined as the trace and determinant of the system matrix A, in terms of the two ratios k/m and c/m.

Ans.

$$\tau = -\frac{c}{m}, \quad \Delta = \frac{k}{m} \tag{3}$$

(c) Consult the $\Delta - \tau$ diagram for classification of fixed points of linear systems as shown in class and in section 5.2 of the textbook. For the following cases, calculate the numerical eigenvalues to identify where on the $\Delta - \tau$ diagram each case lies.

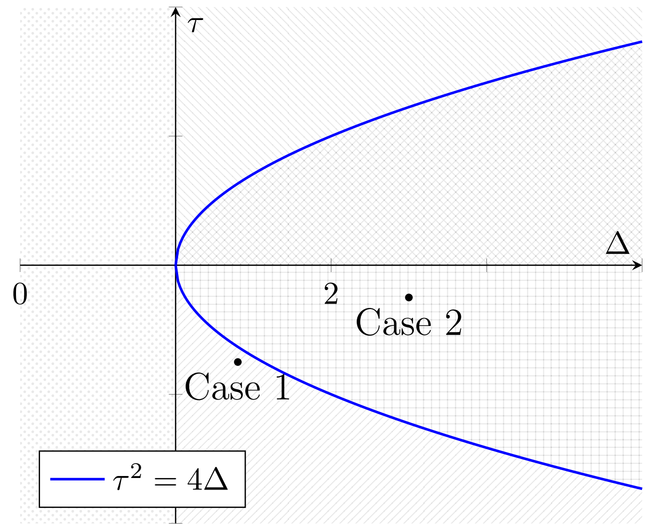
- Case 1: $k = 2$, $m = 5$ and $c = 7.5$ in some appropriate system of units.
- Case 2: $k = 3$, $m = 2$ and $c = 1$ in some appropriate system of units.

Ans. With the convenient representation of τ and Δ above, it seems unnecessary to calculate the eigenvalues themselves; we can classify them directly. We find:

- Case 1: $k = 2$, $m = 5$ and $c = 7.5$: $\boxed{\tau = -7.5/5 = -3/2}$, $\boxed{\Delta = 2/5}$.
- Case 2: $k = 3$, $m = 2$ and $c = 1$: $\boxed{\tau = -1/2}$, $\boxed{\Delta = 3/2}$.

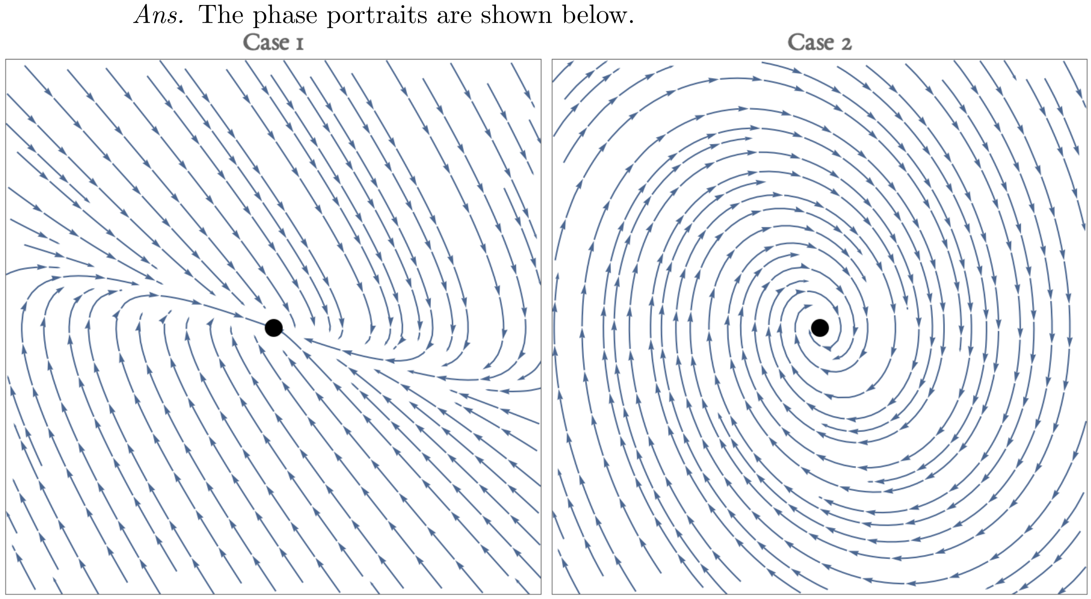
(d) Draw points on the $\Delta - \tau$ diagram shown in fig. 1 to indicate where case 1 and 2 lie.

Ans. The diagram is drawn below.



(e) Make a computer-generated plot of the phase portrait near the fixed point $\boldsymbol{x}^* = [0, 0]^\top$ for each of these cases. Include as much information in this plot as you think is necessary to convey the information it contains.

Ans. The phase portraits are shown below.



(f) Explain the physical significance of the difference between the two regions of the $\Delta - \tau$ diagram in which case 1 and case 2 each fall. How would you explain, in words, to someone the difference between these two cases?

Ans. Case 1 is 'overdamped': no matter what the initial condition, the mass never does more than one back-and-forth motion. Case 2 is still damped, but to a lesser degree; the mass oscillates back and forth past its equilibrium position for a while before it loses steam and it settles at its rest location.

(g) Make physics-based arguments to show that the spring-mass-dashpot system can only fall in a subset of the five regions shown in fig. 1. Identify this subset, i.e., identify which of the five is 'allowed' and which is 'not allowed' based on your physical intuition about this system. In your answer, ignore the 'edge cases' that lie on the boundaries between regions.

Ans. Since the mass, spring constant, and damping coefficient must all be positive, we know from eq. (3) that τ is necessarily negative. Thus, only the bottom half of the diagram is accessible. Similarly, Δ is necessarily positive. Thus, only the bottom-right of the $\tau - \Delta$ plane is physically meaningful.

2. This question is about a general $n = 2$ linear differential equation $\dot{\boldsymbol{x}} = A\boldsymbol{x}$.
Plot phase portraits (near the origin, which is a fixed point) for twelve different linear systems corresponding to the points A through L shown on figure 1. Thus, for each of the twelve cases, you need to find some matrix A that has the properties (trace and determinant) that correspond to the region or curve on which the points are located. You are encouraged to use `pplane` or `Mathematica` to generate these phase portraits. For each of the twelve cases,

- Include an image of the phase portrait, with an appropriate number of trajectories to show the qualitative behavior;
- Write down either the two equations $\dot{x} = f_1(x, y), \dot{y} = f_2(x, y)$ or the matrix A that you used;
- Use section 5.1 and 5.2 of Strogatz's text to classify the fixed point at the origin, if possible.

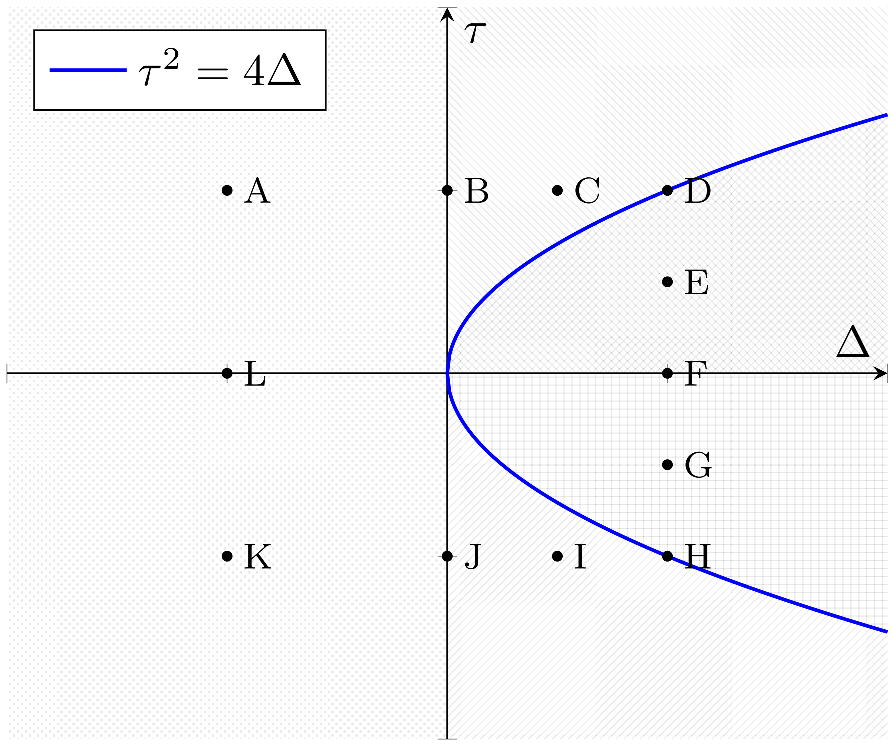
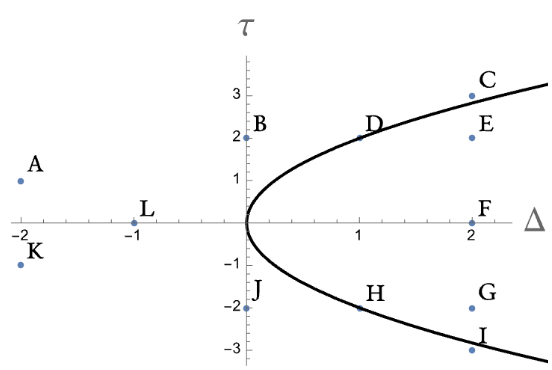
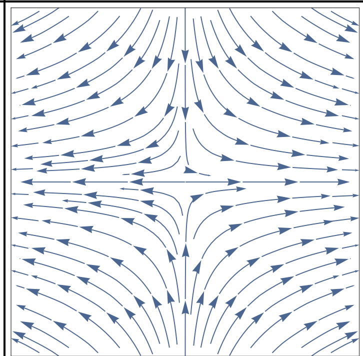
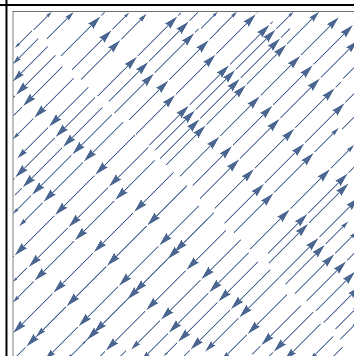
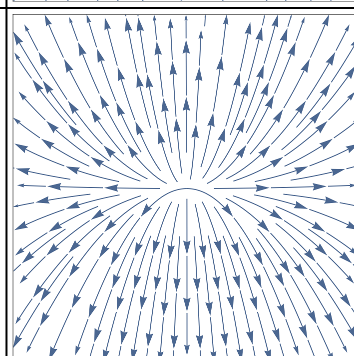
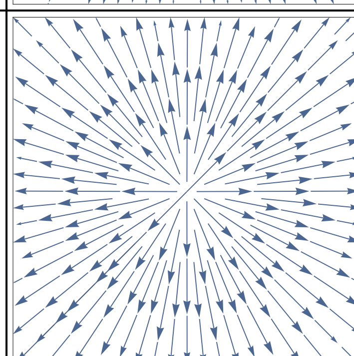
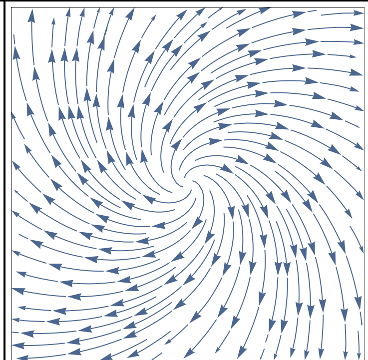
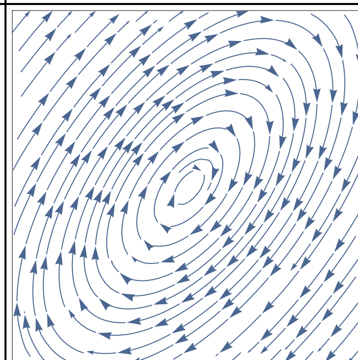
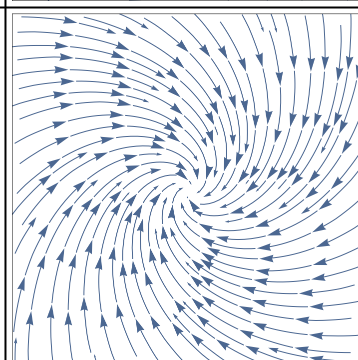
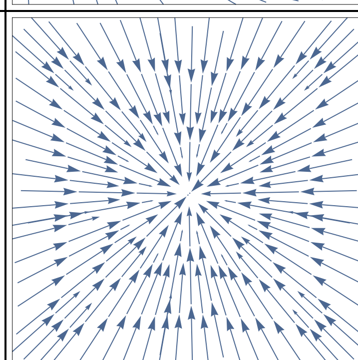


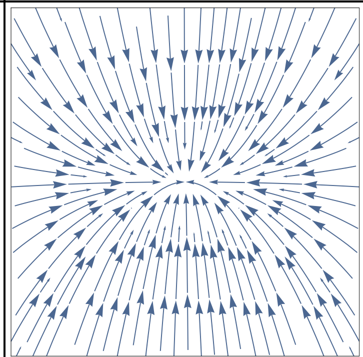
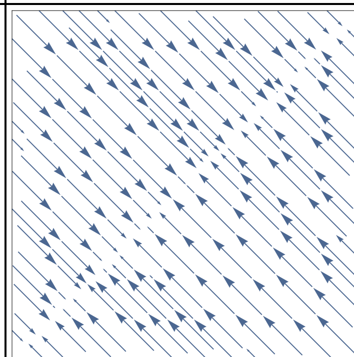
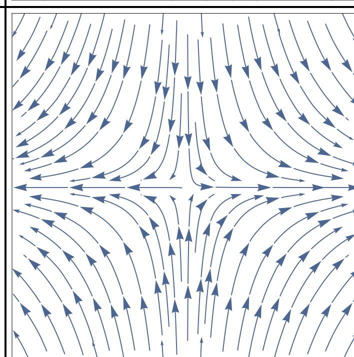
Figure 1: Classification diagram for fixed points of linear systems

Ans. The following points in the $\Delta - \tau$ diagram were chosen.



Fixed Point	Type	Vector Field A x	Phase Portrait
A	Saddle	{2 x, -y}	
B	Line of fixed points	{x + y, x + y}	
C	Unstable node	{x, 2 y}	
D	Star	{x, y}	

Fixed Point	Type	Vector Field A x	Phase Portrait
E	Unstable spiral	{x + y, -x + y}	
F	Center	{-x + 1.5 y, -2 x + y}	
G	Stable spiral	{-x + y, -x - y}	
H	Star	{-x, -y}	

Fixed Point	Type	Vector Field A x	Phase Portrait
I	Stable node	$\{-x, -2y\}$	
J	Line of fixed points	$\{-x+y, x-y\}$	
K	Saddle	$\{x, -2y\}$	
L	Saddle	$\{x, -y\}$