1. Lotka-Volterra Population Dynamics. Consider a system of equations that describes the population of two species undergoing competition:

$$\dot{x} = x\left(5 - \frac{x}{2} - 4y\right) \tag{1a}$$

$$\dot{y} = y\,(5 - x - 3y) \tag{1b}$$

   (a) Generate the nonlinear phase plane for this system using a computer program of your choice. Identify the locations of the fixed points **visually** and classify their type by inspection.

   (b) On a copy of the phase plane, sketch the 'basin of attraction' for all attracting fixed points.

   (c) Determine the 'carrying capacity' of the two species.

   (d) Consider three cases in which the population dynamics are initialized with the following initial values in some appropriate units (e.g., you could think of the numbers as representing thousands of animals).

Case	Rabbits	Sheep
1	0.15	0.08
2	0.16	0.075
3	0.15	0.075

   For each case,

      i. Plot the population of sheep and rabbits on a single set of 'number vs time' axes. There will be six curves in total; use appropriate legends to communicate this information effectively.

      ii. Plot the same information on the phase plane by showing three different trajectories.

2. Consider the two-species population dynamics model shown below.

$$\dot{N}_1 = r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) - b_1 N_1 N_2 \tag{2a}$$

$$\dot{N}_2 = r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) - b_2 N_1 N_2 \tag{2b}$$

   (a) Interpret the constants r_j, K_j, and b_j, $j = 1, 2$ in terms of the population dynamics of each species.

   (b) Assuming that N_1 and N_2 measure a quantity called 'units' that measures population, determine the units of the constants in these equations.

   (c) Show that, with an appropriate re-scaling of N_1, N_2 and time t, eq. (2) can be re-written in non-dimensional form as

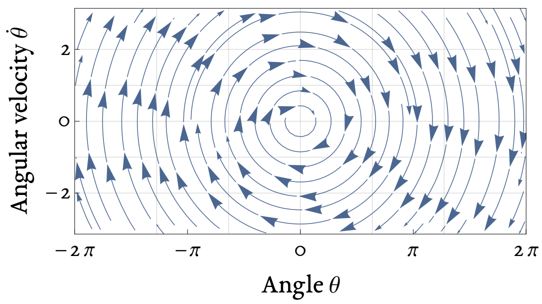
$$\frac{dx}{d\tau} = x(1 - \alpha_1 x) - \frac{xy}{\alpha_2}, \tag{3a}$$

$$\frac{dy}{d\tau} = y(\eta - y) - xy. \tag{3b}$$

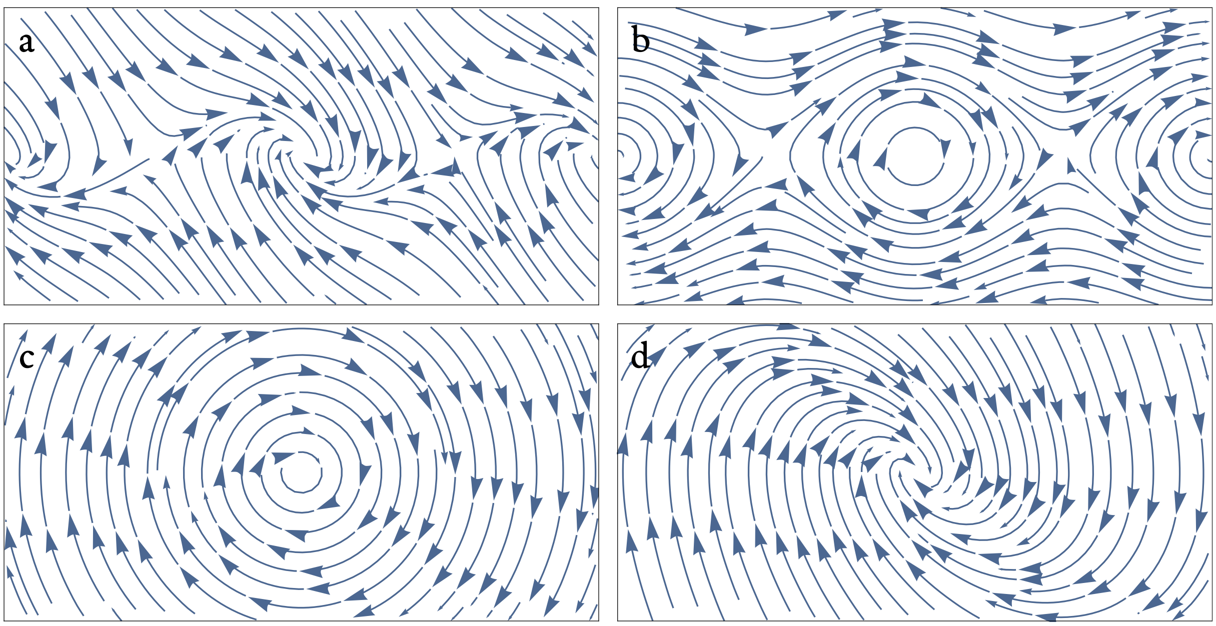
   Find the constants α_1, α_2 and η in terms of the the coefficients from (2), i.e., in terms of r_j, K_j, and b_j for $j = 1, 2$.

   (d) Choose three qualitatively different phase portraits that arise for some choices of α_1, α_2, and η. One of these must be a state of mutual coexistence between rabbits and sheep. Plot the phase portrait for each of these three choices, and write a couple of sentences about the population dynamics shown in each of them.

3. Nonlinear terms and damping. The motion of a pendulum can be succinctly represented using a phase portrait of the kind shown below. In subsequent phase portraits, we will drop the axis labels for neatness.



Now consider the following phase portraits.



(a) Use what you know about the governing equations of a pendulum, and what you learned in Lab 1, to match each of the phase portraits above to one of the physical systems described below. Write a sentence or two explaining your choices, using (simplified, if necessary) equations to support your answers.

1. A 'linear' pendulum without damping
2. A 'nonlinear' pendulum without damping
3. A 'linear' pendulum with damping
4. A 'nonlinear' pendulum with damping

(b) Classify all of the fixed points encountered in the nonlinear systems above. Which of the nonlinear systems' fixed points are **hyperbolic** fixed points?

4. It is known from the second law of thermodynamics that the damping coefficient c in the equation for a harmonic oscillator

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{4}$$

cannot be negative.

   (a) What is the essential feature of the phase portrait of a system with negative damping? You should use a sketch in your answer.

   (b) What are the conditions on τ and Δ (recall that these refer to the trace and determinant of the Jacobian matrix) for the $(0, 0)$ fixed point of an oscillating system such as the one in (4) to be consistent with the second law of thermodynamics? You may wish to think about the $\tau - \Delta$ diagram.

   (c) Express your answer to the above in terms of the eigenvalues λ, i.e., what are the conditions that must be obeyed by the eigenvalues of the Jacobian of the $(0, 0)$ fixed point of (4) such that the second law of thermodynamics is not violated? You may wish to recall that the trace is equal to the sum of the eigenvalues and the determinant is equal to the product of the eigenvalues.

5. Consider a 2-dimensional dynamical system written in polar coordinates instead of the usual x, y coordinates. The two coordinate systems are related by the usual equations

$$x^2 + y^2 = r^2, \text{ and}$$
$$\tan\theta = \frac{y}{x}.$$

The dynamical system is

$$\dot{r} = -r \tag{5a}$$
$$\dot{\theta} = \frac{1}{\log r} \tag{5b}$$

where log refers to the natural logarithm.

   (a) Does this system have a fixed point? If so, where? Answer this question without a computer.

   (b) It can be shown that linearizing the dynamics of eq. (5) leads to a Jacobian matrix of the form $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. What type of fixed point does linearization predict system (5) to have?

   (c) Solve the equations (5) analytically to decide the long-time behavior of r and θ. To do this, you need to find expressions $r(t)$ and $\theta(t)$; your answers will include constants that depend on the initial conditions chosen. It may be convenient to denote the initial conditions with r_0 and θ_0.

   (d) Use your answer to the previous part to determine what type of fixed point this system **really** has. Is this the same as what the linearization predicted?

6. Potentials. Consider the following two systems, which should already be familiar to you.

$$\ddot{\theta} + \sin\theta = 0 \qquad\qquad \ddot{x} + (x - 1) = 0.$$

   (a) For each system, define a potential V that is a function of the position coordinate x or θ. Recall that the potential is defined such that the right hand side of the differential equation above should be equal to (negative of) the derivative of the function V that you choose.

   (b) Show that there exists a conserved quantity E that is a function of position and velocity (i.e., x and $\dot{x}$, or θ and $\dot{\theta}$) for each of these systems. Write this conserved quantity explicitly.

   (c) Plot the potential V against the position coordinate for each of the two systems.

   (d) Where, if any, is this potential a local minimum? Explain physically why the local minimum is at this point.

   (e) Where, if any, is this potential a local maximum? Explain physically why the local maximum is at this point.