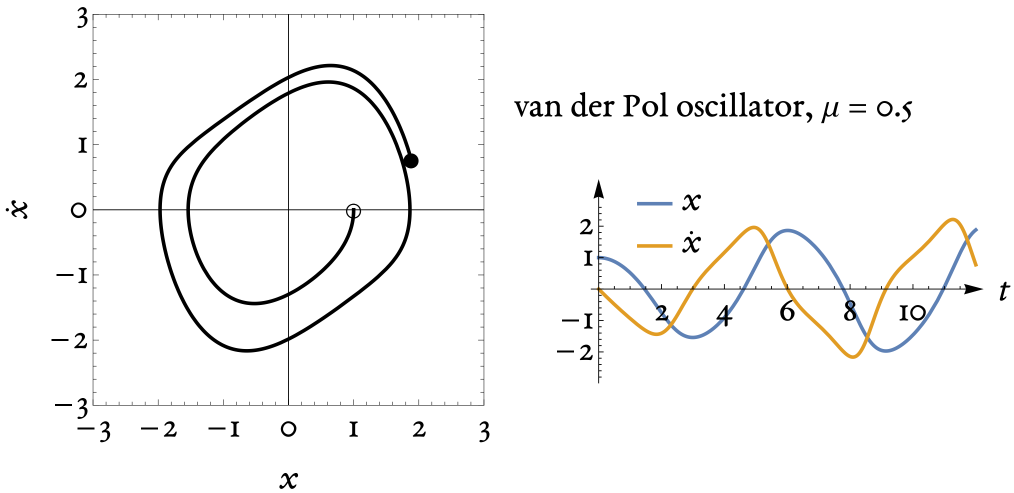
1. The van der Pol oscillator. Consider the van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \tag{1}$$

   (a) Numerically solve (1) using a programming language of your choice for the initial condition

$$x(0) = 1, \quad \dot{x}(0) = 0$$

   with $\mu = 1$. Plot your results in two different forms: (1) parametrically with x on the horizontal axis and $\dot{x}$ on the vertical axis, and (2) both x and $\dot{x}$ against time. Integrate the differential equation(s) from $t = 0$ to $t = 12$. Your figures should look similar to the following. Turn in your code together with the figures for a complete solution.
   Note that the sample has a different value of μ.



   (b) Make a computer-generated phase portrait for this system using a program of your choice; `pplane` is perfectly acceptable. As usual, the horizontal axis should be x and the vertical axis should be $\dot{x}$. Do this for two values of the parameter, $\mu = \{1, 3\}$. The range of the axes should be the same for both cases. Your phase portrait should be legible, and should highlight all the interesting dynamical features of the system, including fixed points and limit cycles, if any.

   (c) Consider the case when $\mu = 0$. Does this system have a limit cycle when $\mu = 0$? Elaborate on your answer using the corresponding phase portrait.

   (d) Show, mathematically and with the help of computer visualization of the phase portrait, that the origin ($x = \dot{x} = 0$) changes its stability when μ changes sign. Recall that, typically, a fixed point is stable if its eigenvalues have negative real part, and unstable if its eigenvalues have positive real part.

2. Consider the system

$$\dot{x} = \mu x + y + \sin x \tag{2a}$$
$$\dot{y} = x - y \tag{2b}$$

   (a) Write down the equations that must be satisfied by the coordinates of the fixed points of this system. You need not solve it at this stage.

   (b) Write down, in symbolic form, the Jacobian for this system in terms of μ and x and/or y.

   (c) Classify the fixed point $(0, 0)$ based on the value taken by μ. What type is it, and for what value of μ does the answer change?

(d) For small distances away from $(0,0)$, some other fixed points may also exist. Determine an approximate expression for the coordinates of these other fixed points in terms of μ. You may need to use the series expansion of $\sin x$ or $\cos x$, as appropriate. Recall that, near $x = 0$,

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$
$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

(e) What type of bifurcation occurs in this system?

3. Applying the Poincare-Bendixson Theorem. Consider the system

$$\dot{x} = a - x - \frac{4xy}{1+x^2} \tag{3a}$$
$$\dot{y} = bx\left(1 - \frac{y}{1+x^2}\right) \tag{3b}$$

where the parameters are set to $a = 10, b = 2$.

(a) Plot the **nullclines** of this system in the range $x, y > 0$. Use this information to determine the location of a fixed point in this quadrant.

(b) Sketch some arrows on the nullclines to indicate the direction of flow on the nullclines. Hint: these will simply be horizontal or vertical arrows.

(c) On the same axes, draw the rectangle defined by the edges of the region

$$\{x, y\} \text{ such that } 0 < x < 10,\ 0 < y < 101.$$

Sketch a few arrows of the vector field along the edges of this rectangle.

(d) Using linearization, determine the type of fixed point that occurs at the point you determined in part 3a. Use your result to show that any trajectory that starts ‘near the fixed point will move away from the fixed point. A mathematical proof is not necessary; words and a sketch is enough.

(e) Use the Poincare-Bendixson Theorem to argue why it must be the case that a limit cycle exists for this system. Indicate, using a sketch or words, the region $\mathcal{D}$ of phase space in which the theorem applies.

4. Complete the in-class exercise from lecture 9.2. You can access the exercise at the following URL: `https://emadmasroor.github.io/classes/E91_S25/Exercises/Exercise5.pdf`