Mon, Mar 3   Lecture 12

When are systems conservative?

↳ Physically, they correspond to frictionless mechanical systems + others

↳ Mathematically, when you can find $E(\underline{x})$.

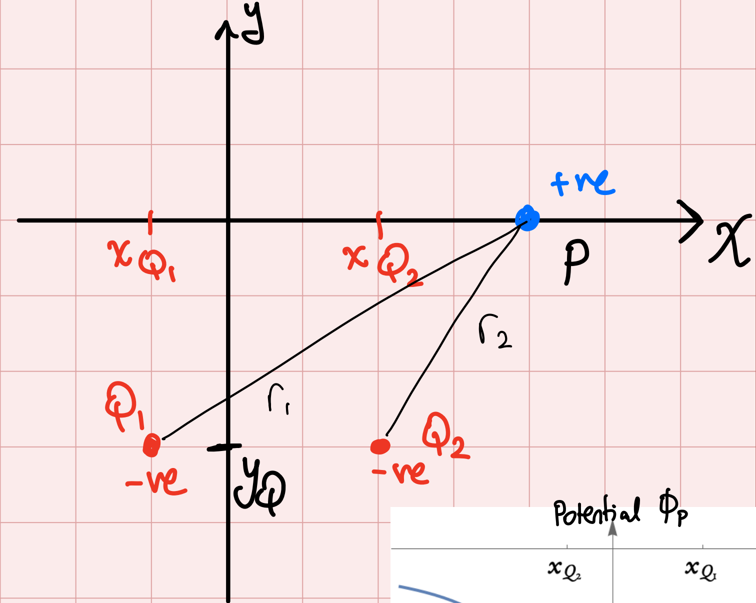
Ex

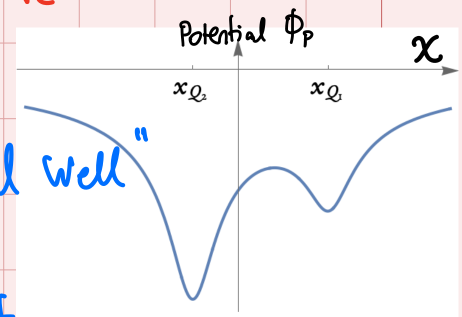
A positive charge P is confined to move on x-axis.

$$r_1 = \sqrt{(x - x_{Q_1})^2 + y_Q^2}$$

Location of P

$$r_2 = \sqrt{(x - x_{Q_2})^2 + y_Q^2}$$





"Potential Well"

Electric Potential

$$\Phi_P = \frac{Q_1}{r_1} + \frac{Q_2}{r_2}$$

$$\text{Force} = -\nabla\phi$$

$$= \frac{Q_1}{r_1^2}\hat{e}_{r_1} + \frac{Q_2}{r_2^2}\hat{e}_{r_2} \rightarrow \text{horz component}$$

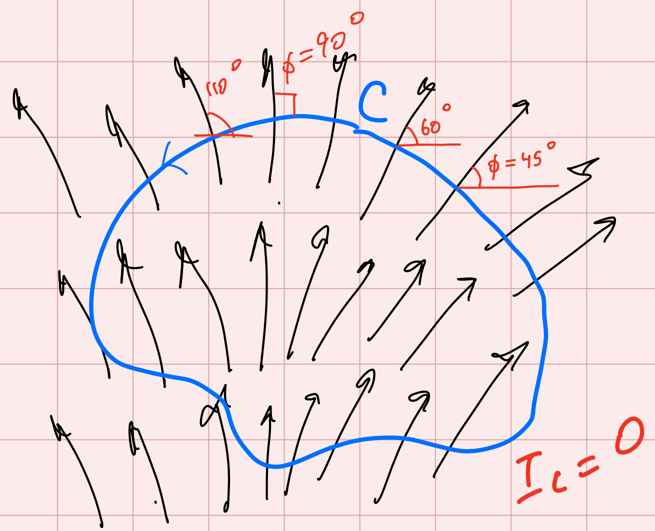
Horz force: $\dfrac{Q_1}{(x-x_{Q_1})^2+y_Q^2}\dfrac{x-x_{Q_1}}{\sqrt{(x-x_{Q_1})^2+y_Q^2}} + \dfrac{Q_2}{(x-x_{Q_2})^2+y_Q^2}\dfrac{x-x_{Q_2}}{\sqrt{(\dots)}}$

" $\ddot{x}$ = force "

Index of a closed curve in a vector field.

A measure of the "winding" of the vector field.

Let $\underbrace{\dot{\underline{x}} = f(\underset{\sim}{x})}_{\text{a smooth vector field}}$, $\underset{\sim}{x} \in \mathbb{R}^2$ and C a closed curve (does not self-intersect) (does not pass thru fixed pts of the vector field)



Let ϕ be the angle between the vector field $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}$ and the horizontal for any point on C.

$\Rightarrow \phi = \tan^{-1}(\dot{y}/\dot{x})$

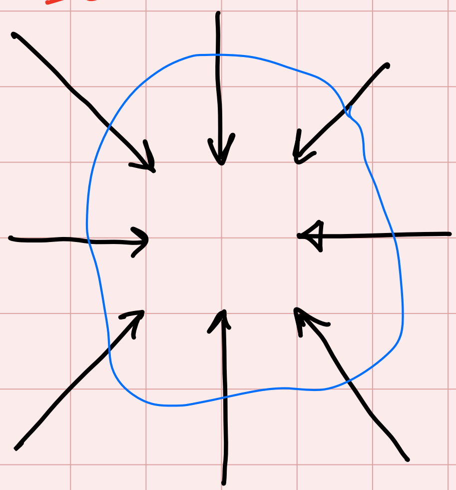
$[\phi]_C$: net change in ϕ over one c.c.w. loop around C.

Index of C is:

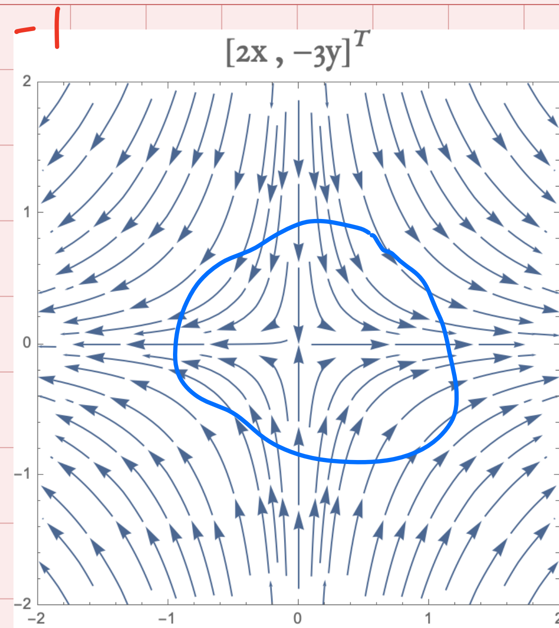
$$I_C = \frac{1}{2\pi}[\phi]_C$$

net number of counter-clockwise revolutions made by the vector field $f(\underset{\sim}{x})$ as $\underset{\sim}{x}$ moves counterclockwise around C.

$I_c = +1$



$I_c = -1$



spirals, centers, stars, degenerate nodes : $I_c = +1$

Saddles $I_c = -1$

- $I_c = 0$ if no fixed pts inside C.
- If C is a trajectory, $I_c = +1$
- If C can be continuously deformed into C' without passing thru a fixed pt, $I_c = I_{c'}$.
- If all arrows in the vector field reverse direction $\vec{f} \to -\vec{f}$ then I does not change.
- The index of a fixed point $= I_c$ for any C that encloses only that fixed pt.
- Any closed orbit in phase plane must enclose fixed pts whose indices sum up to $+1$.
- If C surrounds multiple fixed pts, $I_c = \sum$ index of each fixed pt.