Proving the absence of closed orbits

closed orbits: a superset of limit cycles.

1) Gradient Systems approach.

If a system $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x})$ can be written as $\dot{\underset{\sim}{x}} = -\nabla V(\underset{\sim}{x})$ for some scalar function $V(\underset{\sim}{x})$, then no closed orbits exist for this system.

V: continuously differentiable, single-valued.

Recall: in 1-d, $\dot{x} = f(x)$, we could always find $V(x)$ such that $f(x) = -dV/dx$

no oscillations in 1-d.

in 2-d, not so sure; hard to find V

$$\ddot{\theta} + \sin\theta = 0$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\sin x_1$$

$$\underset{\sim}{f}(\underset{\sim}{x}) = \begin{bmatrix} x_2 \\ -\sin x_1 \end{bmatrix}$$

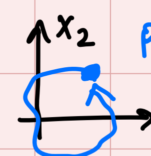
no V can be found.

look for V, s.t.

$$\underset{\sim}{f} = -\nabla V \qquad -\frac{\partial V}{\partial x_1} = x_2 \qquad -\frac{\partial V}{\partial x_2} = -\sin x_1$$

cannot establish the absence of closed orbits.

Proof $\dot{\underset{\sim}{x}} = -\nabla V(\underset{\sim}{x})$



change in V from $t=0$ to $t=T$. → must be 0.

$$\Delta V = \int_0^T dV = \int_0^T \frac{dV}{dt} \cdot dt$$

$$\frac{dV}{dt} \to \underbrace{\frac{dV}{d\underset{\sim}{x}} \cdot \frac{d\underset{\sim}{x}}{dt}}_{(\nabla V) \cdot \dot{\underset{\sim}{x}}}$$

$$\int_0^T (\nabla V \cdot \dot{\underset{\sim}{x}})\, dt$$

$$= \int_0^T (-\dot{\underset{\sim}{x}} \cdot \dot{\underset{\sim}{x}})\, dt = \underbrace{-\int_0^T \|\dot{\underset{\sim}{x}}\|^2 dt}_{<0} \quad \text{CONTRAD.}$$

Given a system $\dot{\underset{\sim}{x}} = -\nabla V(\underset{\sim}{x})$ a closed orbit is impossible

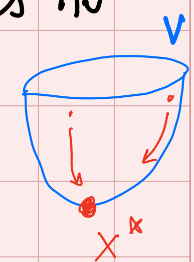
2) Liapunov Function approach

$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ with $f(\underset{\sim}{x}^*) = \underset{\sim}{0}$

If we can find a continuously differentiable real-valued function $V(\underset{\sim}{x})$ that:

$$\left.\begin{array}{ll} V(\underset{\sim}{x}) > 0 & \forall\ \underset{\sim}{x} \neq \underset{\sim}{x}^* \\ V(\underset{\sim}{x}^*) = 0 & \\ \dot{V}(\underset{\sim}{x}) < 0 & \forall\ \underset{\sim}{x} \neq \underset{\sim}{x}^* \end{array}\right\}$$

then system has no closed orbits.



where $V(\underset{\sim}{x})$: "an energy-like function that decreases along trajectories."

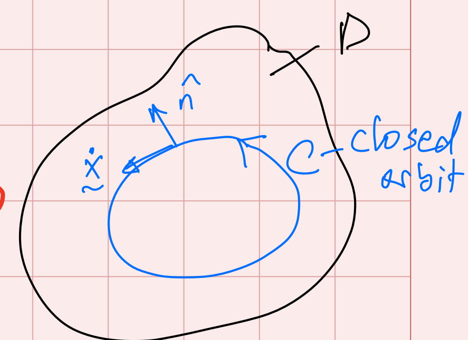
3) Dulac's Criterion

$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ and f is defined on $D \subset \mathbb{R}^2$: a simply connected subset.

If there exists $g(\underset{\sim}{x})$ such that $\nabla \cdot (g\dot{\underset{\sim}{x}})$ has one sign throughout D, then there are no closed orbits entirely within D.

$$\underbrace{\iint_A \underbrace{\nabla \cdot (g\dot{\underset{\sim}{x}})}_{\text{"has one sign"}}\, dA}_{\Rightarrow \text{ term } \neq 0} = \underbrace{\oint g\dot{\underset{\sim}{x}} \cdot \hat{n}\, ds}_{=0} \qquad \dot{\underset{\sim}{x}} \cdot \hat{n} = 0$$

$\Rightarrow$ No C exists



Presence of Closed orbits

Poincaré–Bendixson Thm

If

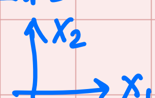
1. D is a closed, bounded subset of $\mathbb{R}^2$.
2. $\dot{\underline{x}} = f(\underline{x})$ is defined on some open set that includes D.
3. D does not have any fixed pts.
4. There exists a trajectory that is confined in D: starts in D and stays inside D for all future time

Then "trapping region"

C is a closed orbit or approaches a closed orbit.

implications for Chaos

The topology of $\mathbb{R}^2$ prevents anything too wild from happening in the phase plane



we know that trajectories of $\dot{\underline{x}} = f(\underline{x})$ cannot self-intersect.

If a trajectory is known to be trapped in a certain finite subset of $\mathbb{R}^2$, it must eventually settle down into a limit cycle.

It cannot keep wandering forever : it will eventually run out of room.

In $n \geq 3$ autonomous systems, chaos is possible because $\mathbb{R}^3$ has infinitely more room than $\mathbb{R}^2$. If $n \geq 3$, a trajectory can be confined to a subset of $\mathbb{R}^n$ and wander around forever without self intersecting. ⁓ STRANGE ATTRACTORS.