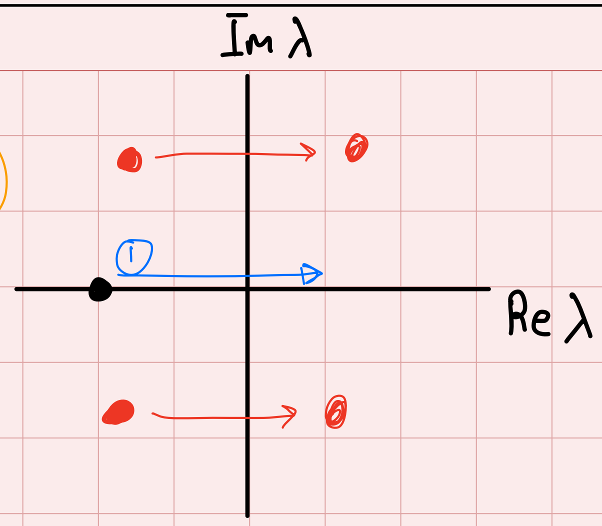
Wed, Mar 26 Lecture 16

Hopf Bifurcations (Limit cycles)



① A stable node becomes unstable

Bifurcations seen so far involve $\lambda = 0$ at the critical point.

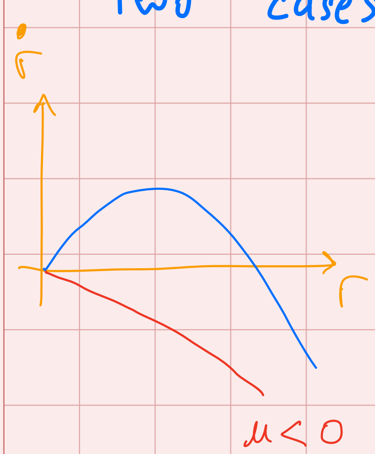
② Hopf Bifurcation

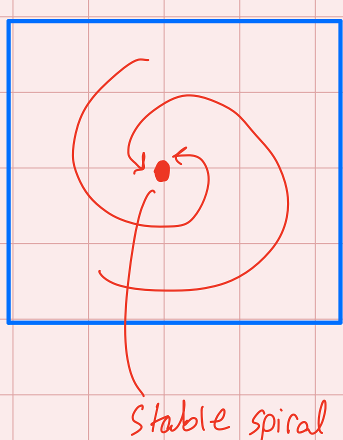
Bifurcations occur when $\mathrm{Im}(\lambda) \neq 0$

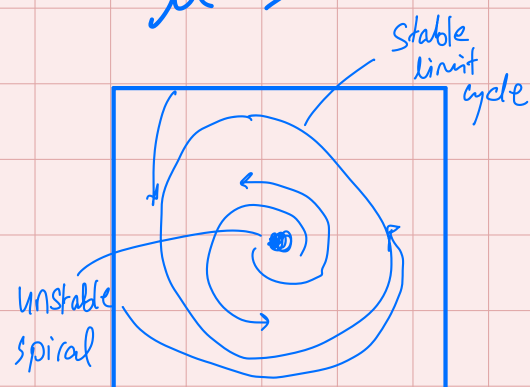
$$\dot{r} = \mu r - r^3$$
$$\dot{\theta} = \omega + b r^2$$

$\longrightarrow$ fixed points in r occur at $r = 0$, $r = \pm\sqrt{\mu}$ (only if μ is positive)

Two cases: $\mu < 0$ $\mu > 0$







convert to x, y

$$\dot{r} = \mu r - r^3$$
$$\dot{\theta} = \omega + br^2$$

$x = r\cos\theta$, $y = r\sin\theta$

$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$ plug in

$$\dot{x} = (\mu r - r^3)\cos\theta - r(\omega + br^2)\sin\theta$$
$$= \mu x - (x^2+y^2)x - y(\omega + b(x^2+y^2))$$
$$= \mu x - \omega y + \text{higher-order terms}$$

$\dot{y} = \omega x + \mu y + \dots$ $\Rightarrow$ $A = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}$

Eigenvalues are $\boxed{\lambda = \mu \pm i\omega}$

As $\mu \uparrow$, eigenvalues cross the imaginary axis.

Types of Hopf Bifurcation:

supercritical
subcritical
saddle-node
∞-period
homoclinic

<in-class exercise>