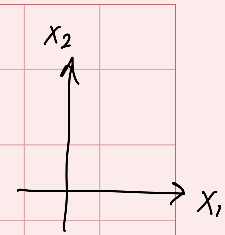
Quasiperiodicity

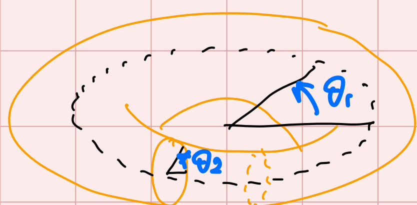
Mon, Mar 31 Lecture 17

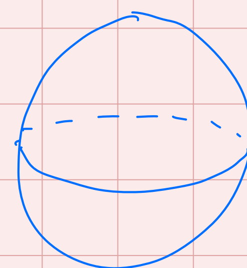


We usually think of $n=2$ dynamics on $\mathbb{R}^2$

i.e $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}$

But if the two variables can be interpreted as angles then the phase space is not the plane but is instead the surface of a torus

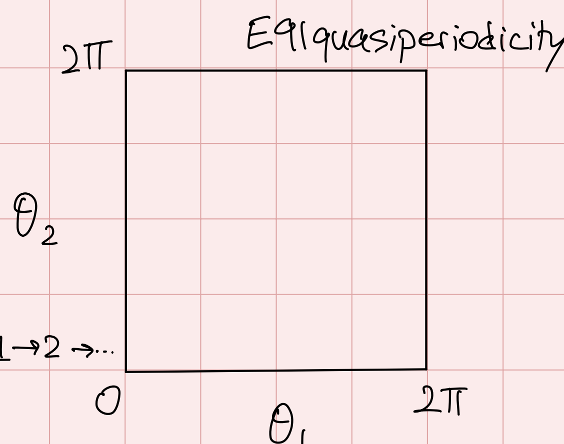


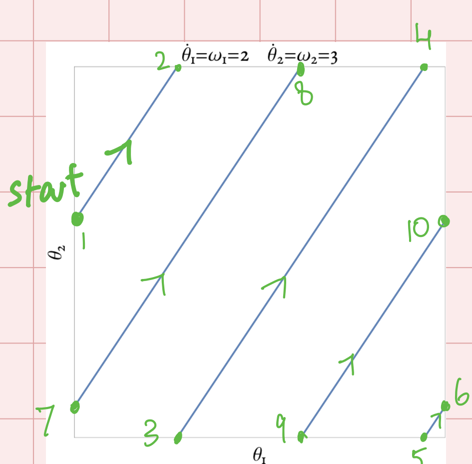


Simple $n=2$ system with periodic coordinates

$$\left.\begin{aligned}\dot{\theta}_1 &= \omega_1 \\ \dot{\theta}_2 &= \omega_2\end{aligned}\right\} \text{ for some constant } \omega\text{'s.}$$

Plot trajectories for different ω's.

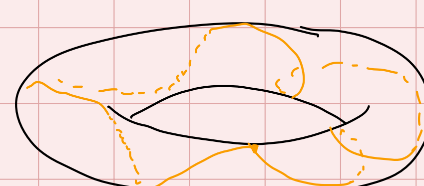




$1 \to 2 \to \cdots \to 10 \to 1 \to 2 \to \cdots$

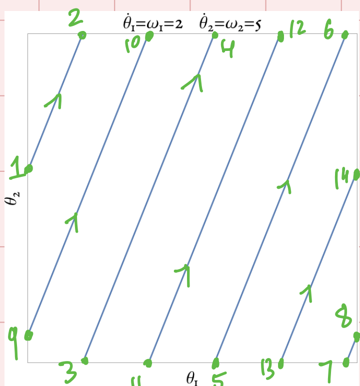
Revolutions in θ_1 :

Revolutions in θ_2 :



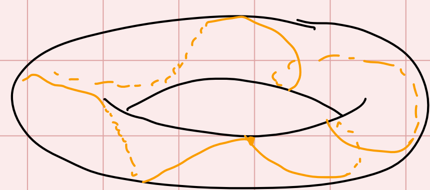
$\omega_1 = 2$ Revolutions in θ_1 : 1

$\omega_2 = 6$ Revolutions in θ_2 : 3



Rev's in θ_1 : 2

Rev's in θ_2 : 5



Trajectory goes twice around the large circle, 5x around small circle, comes back to starting point.

These were examples of periodic flow on torus.

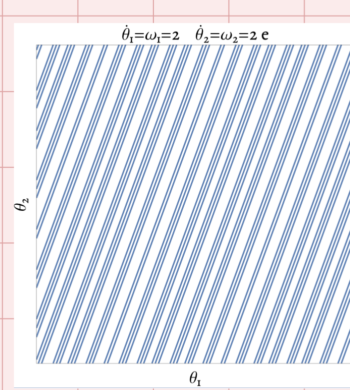
Trajectories are straight lines with slope ω_2/ω_1.

if $\omega_1/\omega_2 = p/q$ for some integers p, q

then θ_1 completes p revolutions in the time θ_2 completes q revolutions.

if ω_1/ω_2 irrational, flow in phase space is quasiperiodic; any trajectory fills the phase space without ever repeating.

tinyurl.com/E91quasiperiodicity



The Lorenz equations

$\sigma, r, b > 0$

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - \underline{xz}$$
$$\dot{z} = \underline{xy} - bz$$

Simplified model of atmospheric convection.

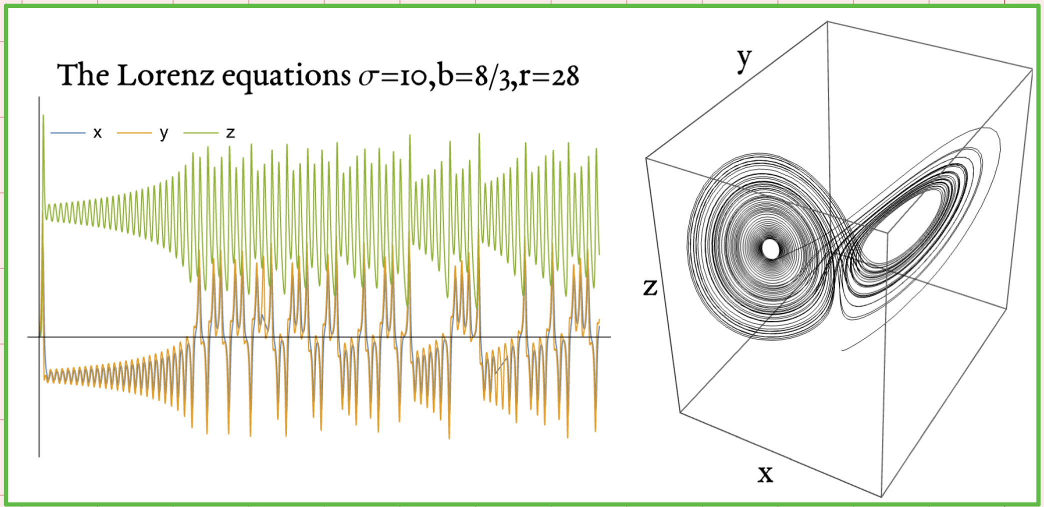
→ Nonlinear

→ 3-dimensional phase space

→ Symmetric : $(x, y, z) \longmapsto (-x, -y, z)$

tinyurl.com/E91lorenz1
tinyurl.com/E91lorenz2

Wed, Apr 2 Lecture 18



→ Dissipative : "Volumes in phase space" shrink exponentially with time

i.e. $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x})$, $\nabla \cdot \underset{\sim}{f} < 0$ for dissipative systems.