$$\begin{cases} \dot{x} = f(x, t) \\ x(0) = x_0 \end{cases} \quad (*)$$

$x \in \mathbb{R}^n$

Chaos is impossible if $n < 3$

To "solve" this IVP means to find a function $x(t)$ that satisfies $(*)$.

- analytical solution : use MATH
- numerical solution : use computer

A solution may exist for all $t \in \mathbb{R}$ or for a subset of $\mathbb{R}$

Mon, Jan 27 Lecture 2

$\dot{x} = f(x, t)$: first-order

$\ddot{x} = f(x, \dot{x}, t)$: 2^{nd}

$\dddot{x} = f(x, \dot{x}, \ddot{x}, t)$: 3^{rd}

e.g. $x(0) = \ldots$

$\dot{x}(0) = \ldots$

$\ddot{x}(0) = \ldots$

$\ddot{x} = f(x, \dot{x})$: autonomous

$\ddot{x} = f(x, \dot{x}, t)$: nonautonomous

Equivalence of n^{th} order differential equations and a system of n 1^{st} order " "

$$\frac{d^n x}{dt^n} = f\left(x, \dot{x}, \ddot{x}, x^{(3)}, x^{(4)}, \ldots, x^{(n-1)}\right)$$

it is always possible to write an equivalent system of n 1^{st} order equations:

Define $y \in \mathbb{R}^n$

$$\begin{aligned}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= y_3 \\
\dot{y}_3 &= y_4 \\
&\vdots \\
\dot{y}_{n-1} &= y_n \\
\dot{y}_n &= f(y_1, y_2, y_3, \ldots)
\end{aligned}$$

Any dynamics problem can be written as

$$\dot{\vec{x}} = f(\vec{x}, t) \qquad \text{where} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \dot{\vec{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n$$
$$x(0) = x_0$$

is linear if it can be written as $\dot{x} = A x$ (A: $n \times n$)

$n=1$, linear $\longrightarrow$ increasing n

$\downarrow$

increasing nonlinearity

chaos lives here

$n \gg 1$, nonlinear

$n = 1$, nonlinear

$$\dot{x} = \sin x, \quad x(0) = x_0 \longrightarrow$$

analytical : $x(t)$ function

numerical : $\{t_i, x_i\}$

geometric

① Analytical

$$\frac{dx}{dt} = \sin x$$

$$\int \operatorname{cosec} x \, dx = \int dt$$

use $t=0$, $x = x_0$ to find C

$$-\log \left| \operatorname{cosec} x + \cot x \right| + C = t$$

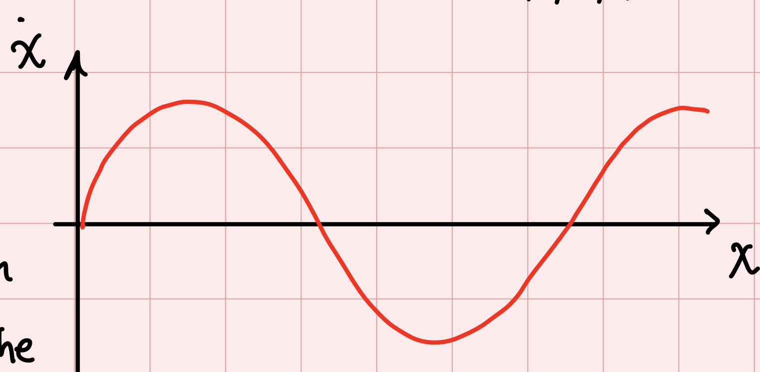
$$\log \left| \frac{\operatorname{cosec} x_0 + \cot x_0}{\operatorname{cosec} x + \cot x} \right| = t$$

2 Numerical $\quad \dfrac{dx}{dt} = \sin x$

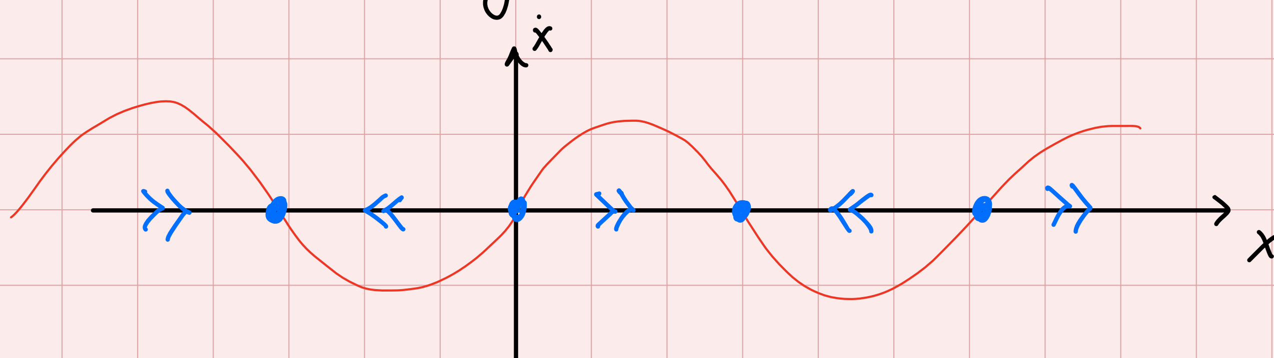
$$\frac{x_{n+1} - x_n}{\Delta t} = \sin x_n \Rightarrow x_{n+1} = \Delta t \sin(x_n) + x_n$$

$n = 0, 1, 2, 3, \ldots$

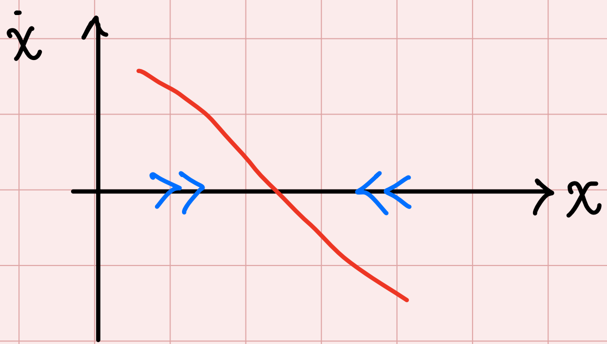
3 Geometric



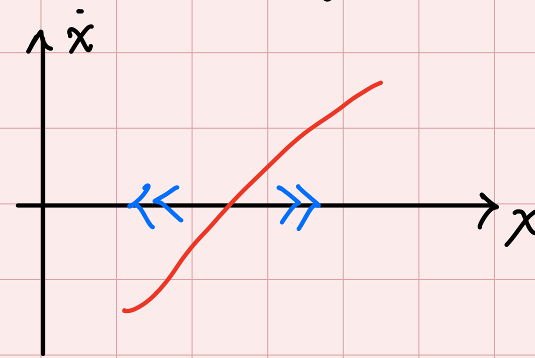
- The state of system is a point on the x-axis
- $\dot{x} > 0$: moving to the right
  $\dot{x} < 0$: Moving to the left. } call this "flow"



Two kinds of "fixed points" emerge:

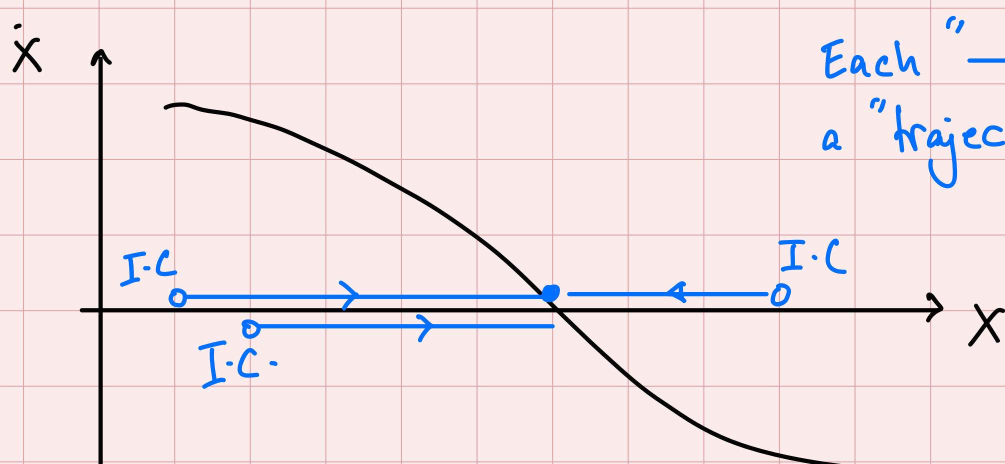


ATTRACTOR
SINK
(stable)



REPELLER
SOURCE
(unstable)

Phase Portrait (for 1-d systems)



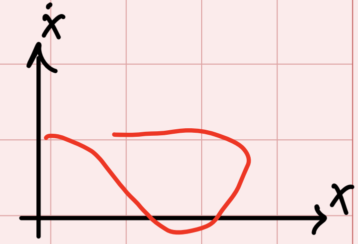
Each "$\longrightarrow$" is a "trajectory".

$\boxed{\text{System } \dot{x}=f(x)}$

- one phase portrait
- (a few ?) fixed points
- infinite trajectories

A diagram showing
- all qualitatively different trajectories
- all fixed points

Note : $f(x)$ must be a function
in addition, we will work with $[f(x)]$'s that are "nice"
= sufficiently smooth.



not allowed

Example $\dot{N} = rN\left[1 - \frac{N}{K}\right]$ Logistic Equation