Mon, Feb 10 Lecture 6

Show that, with appropriate non-dimensionalization,

$$\dot{u} = a u + b u^3 - c u^5$$ is equivalent to

$$\dot{x} = r x + x^3 - x^5.$$ where $\dot{x} = \frac{dx}{d\tau}$

$$U = \sqrt{b/c}$$
$$T = c/b^2$$
$$r = ac/b^2$$

$$\left\{ \begin{array}{l} x = u/U \\ \tau = t/T \end{array} \right.$$

Dynamics with $\underline{n = 2}$

$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}) \qquad \underset{\sim}{x} \in \mathbb{R}^2$$

<u>Conventions</u> the state can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ or } \begin{bmatrix} x \\ y \end{bmatrix} \text{ or } \underset{\sim}{x}$$

<u>Note</u> : f is a vector-valued function of a vector argument.

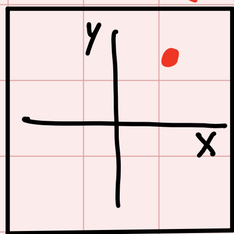
1-dimensional state

only one variable determines state

change : $\dot{x}$

2-dimensional state

(x,y) — determines state



change: $\dot{x}, \dot{y}$

The solutions of $\left\{ \dot{\underset{\sim}{x}} = f(\underset{\sim}{x}),\ \underset{\sim}{x} \in \mathbb{R}^2,\ \underset{\sim}{x}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right\}$

can be visualized as trajectories on the <u>phase plane</u>

$$\ddot{x} = -\omega^2 x \qquad \begin{matrix} \dot{x} \\ \ddot{x} \end{matrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

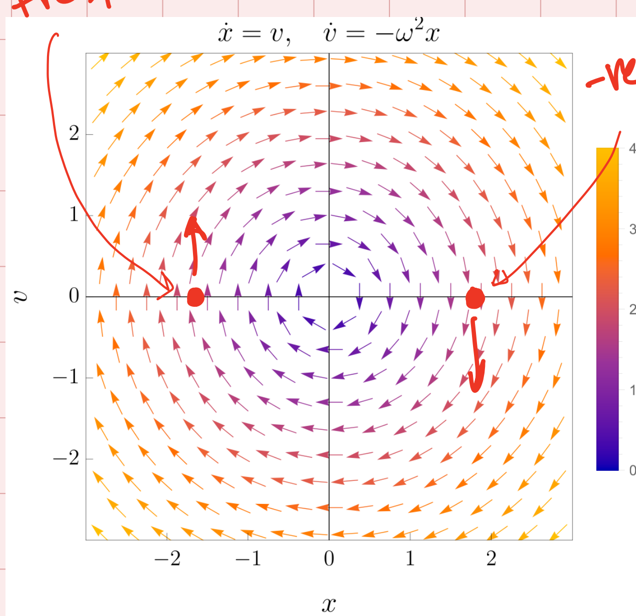
state: x, v

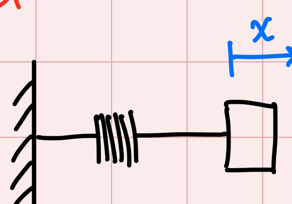
position velocity

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ -\omega^2 x \end{bmatrix}$$

+ve speed

-ve speed





→ $\dot{\underset{\sim}{x}}$ is a vector with 2 components, defined for any point (x,y) on the plane.
i.e. $\dot{\underset{\sim}{x}}$ is a vector field

→ Trajectories are $\{x_1(t), x_2(t)\}$ functions of time parameterized by the initial condition.

or, numerically, ordered pairs parameterized by t.

→ Trajectories are everywhere tangent to vector field.

---

if $f(\underset{\sim}{x})$ is linear, without loss of generality we can express $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ as:

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

<u>Note</u>

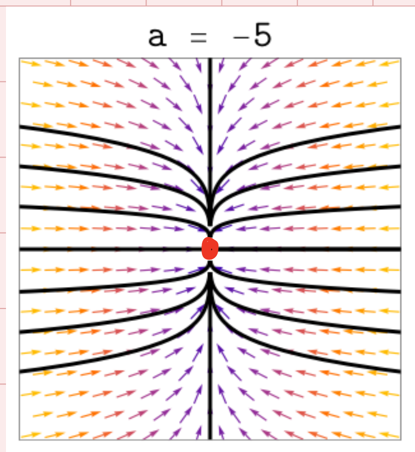
$\underset{\sim}{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is always a fixed pt

study a particular linear system:

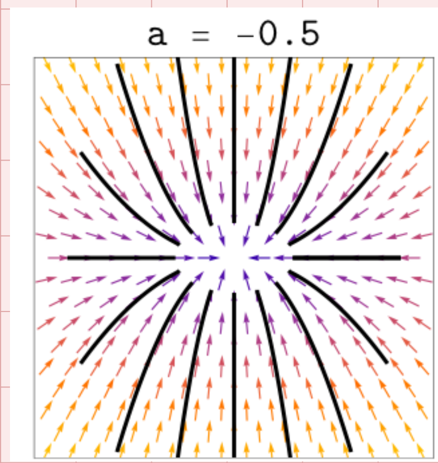
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{aligned} \dot{x} &= ax \\ \dot{y} &= -y \end{aligned} \longrightarrow \begin{aligned} x(t) &= x_0 e^{at} \\ y(t) &= y_0 e^{-t} \end{aligned}$$

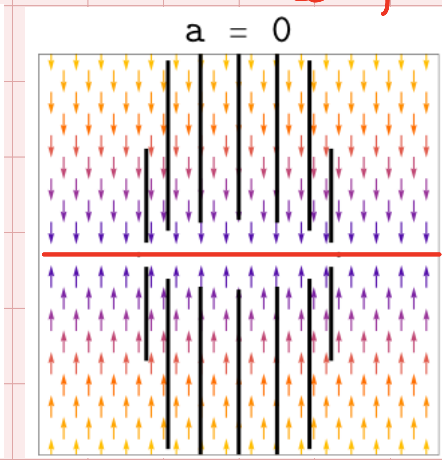
stable node

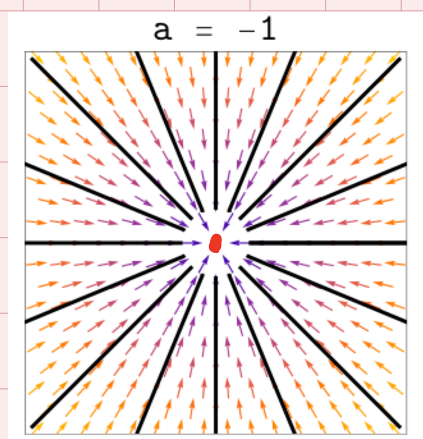


stable node ?

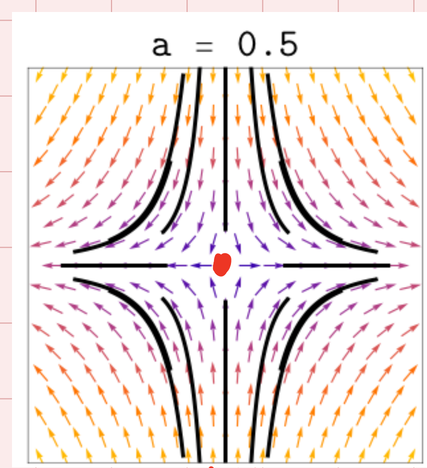


line of fixed pts





symmetric node (star)



saddle point