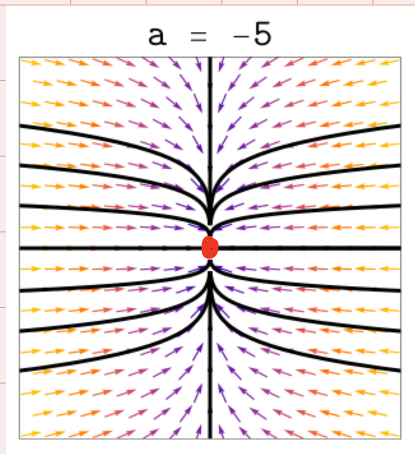
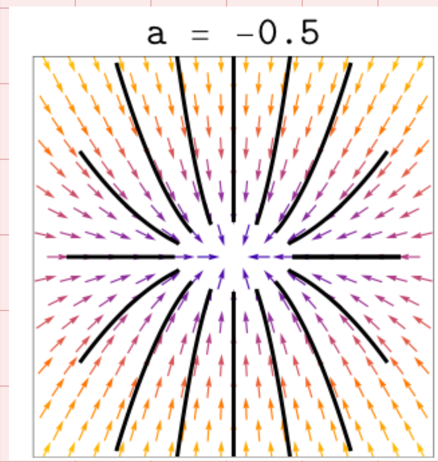
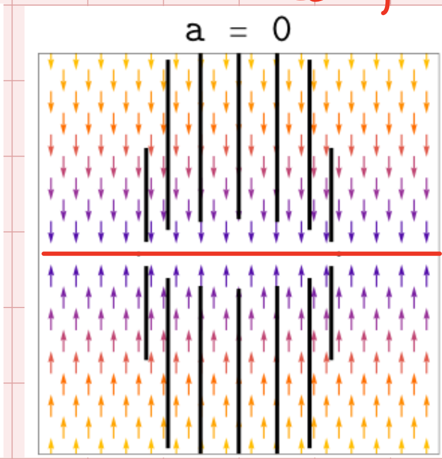
stable node

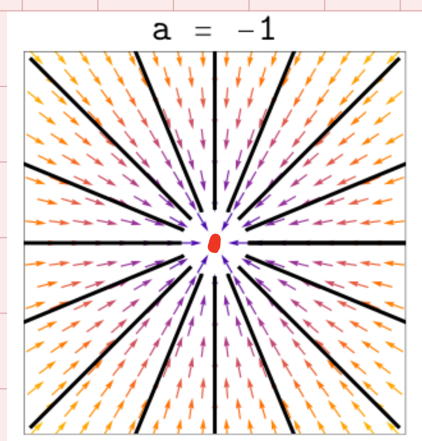


stable node

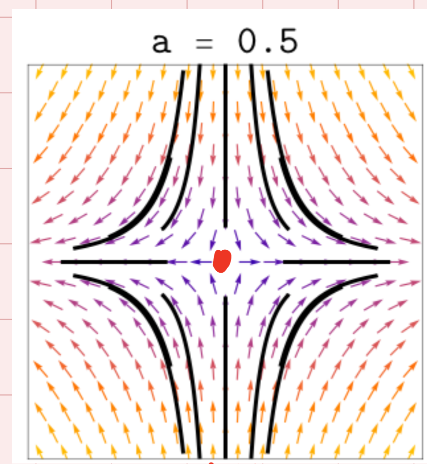


line of fixed pts





symmetric node (star)



saddle point

Wed, Feb 12, Lecture 7

General $n=2$ Linear Systems $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \overbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}^{A} \begin{bmatrix} x \\ y \end{bmatrix}$

Recall that, for $A = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix}$, the x and y axes were $\underbrace{\text{special directions}}_{\text{related to eigenvectors of } A.}$ with $\underbrace{\text{straight-line}}$ trajectories

→ exponential in time

Are there such trajectories for general A ?
What directions do those trajectories travel in ?
i.e. is there a vector $\underset{\sim}{v}$ and λ such that

$$\underset{\sim}{x}(t) = e^{\lambda t}\underset{\sim}{v} \quad ? \qquad \dot{\underset{\sim}{x}} = A\underset{\sim}{x}$$

$$\lambda \cancel{e^{\lambda t}} \underset{\sim}{v} = A \cancel{e^{\lambda t}} \underset{\sim}{v}$$

$$\lambda \underset{\sim}{v} = A \underset{\sim}{v}$$

: λ eigenvalues of A
$\underset{\sim}{v}$ eigenvectors of A.

$$\det(A - \lambda I) = 0 \qquad A - \lambda I = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$

$$(a-\lambda)(d-\lambda) - cb = 0$$

$$\lambda^2 - \lambda a - \lambda d + ad - bc = 0$$

$$\lambda^2 - \underbrace{(a+d)}_{\mathrm{tr}(A),\ \tau}\lambda + \underbrace{ad - bc}_{\det(A),\ \Delta} = 0$$

tr: trace
det: determinant.

$$\boxed{\lambda^2 - \tau\lambda + \Delta = 0}$$

Equation for eigenvalues of A.

$$\boxed{\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}}$$

Note : once you know λ, it is straightforward to calculate $\underset{\sim}{v}$ by solving $\lambda\underset{\sim}{v} = A\underset{\sim}{v}$ for the two components of $\underset{\sim}{v}$.

As long as $\lambda_1 \neq \lambda_2$, any state of the system $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ can be written as a linear combination of the eigenvectors $\underset{\sim}{v_1}$ and $\underset{\sim}{v_2}$.

$$\underset{\sim}{x} = a_1 \underset{\sim}{v_1} + a_2 \underset{\sim}{v_2} \quad \text{for scalars } a_1 \text{ and } a_2.$$

$$\underset{\sim}{x}(t) = (a_1(t))\underset{\sim}{v_1} + (a_2(t))\underset{\sim}{v_2}$$

As $\underset{\sim}{x}$ varies over time, these scalars vary exponentially in time.

and it's possible to write a general solution $\underset{\sim}{x}(t)$ for the differential equation $\dot{\underset{\sim}{x}} = A \underset{\sim}{x}$.

$$\underset{\sim}{x}(t) = c_1 e^{\lambda_1 t} \underset{\sim}{v_1} + c_2 e^{\lambda_2 t} \underset{\sim}{v_2}$$

→ λ's, $\underset{\sim}{v}$'s are eigenvalues and eigenvectors.

→ c's are const. coefficients that depend on initial condition $\underset{\sim}{x}(0)$.

No such general solution exists for $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ if f is not linear in $\underset{\sim}{x}$.

Exercise **Solve**

$$\dot{x} = x + y \qquad x_0 = 2$$
$$\dot{y} = 4x - 2y \qquad y_0 = -3$$

$\dot{x} = Ax$ with $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$. $\quad \tau = -1$, $\Delta = -6$

First, find eigenvalues.

$$\lambda^2 + \lambda - 6 = 0 \Rightarrow \lambda^2 + 3\lambda - 2\lambda - 6 = 0$$
$$(\lambda+3)(\lambda-2) = 0$$
$$\Rightarrow \lambda = 2, -3$$

Then find eigenvectors.

$$\underset{\sim}{A}\,\underset{\sim}{u} = \underbrace{\lambda}_{=2}\,\underset{\sim}{u}$$

$$\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2u_1 \\ 2u_2 \end{bmatrix} \Rightarrow \begin{aligned} u_1 + u_2 &= 2u_1 \\ 4u_1 - 2u_2 &= 2u_2 \end{aligned} \Rightarrow \begin{aligned} u_1 &= 1 \\ u_2 &= 1 \end{aligned}$$

$$\Rightarrow \underset{\sim}{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

one eigenvector, associated with $\lambda = 2$

similarly .... $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$ is the 2^{nd} eigenvector associated with $\lambda = -3$

$$\underset{\sim}{x}(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

at $t=0$, $\underset{\sim}{x} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

at $t=0$

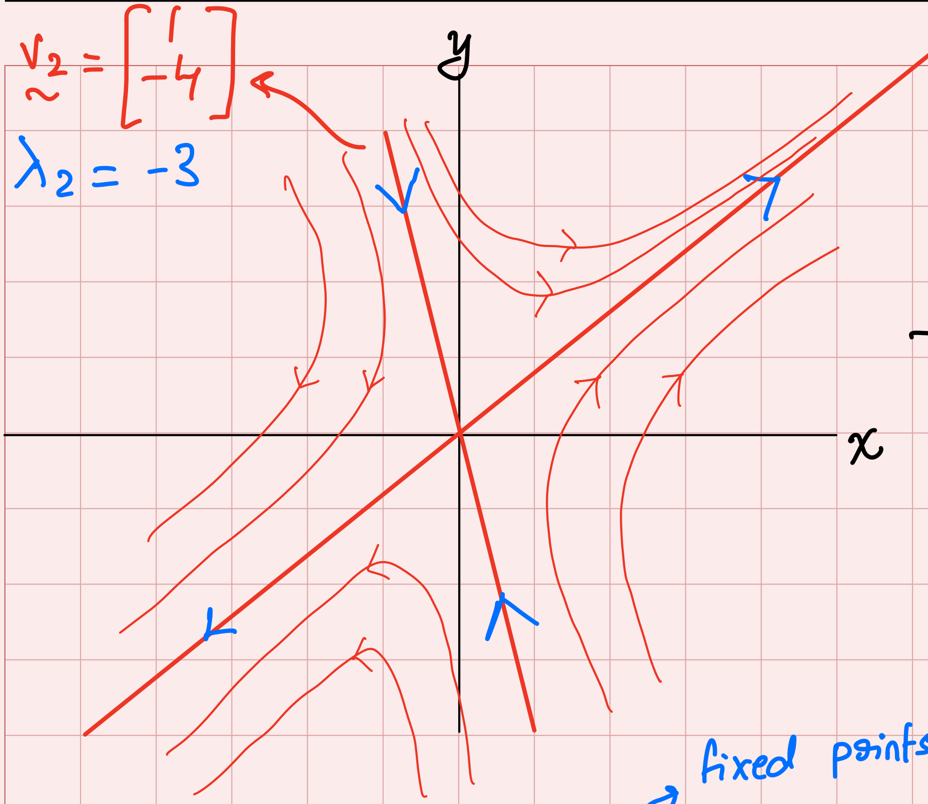
$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} = c_1 e^{0}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{0}\begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$c_1 + c_2 = 2$$
$$c_1 - 4c_2 = -3$$
$$\Rightarrow c_1 = 1, \quad c_2 = 1$$

$$\underset{\sim}{x}(t) = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t}\begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$\underset{\sim}{v}_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

$\lambda_2 = -3$



$\underset{\sim}{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

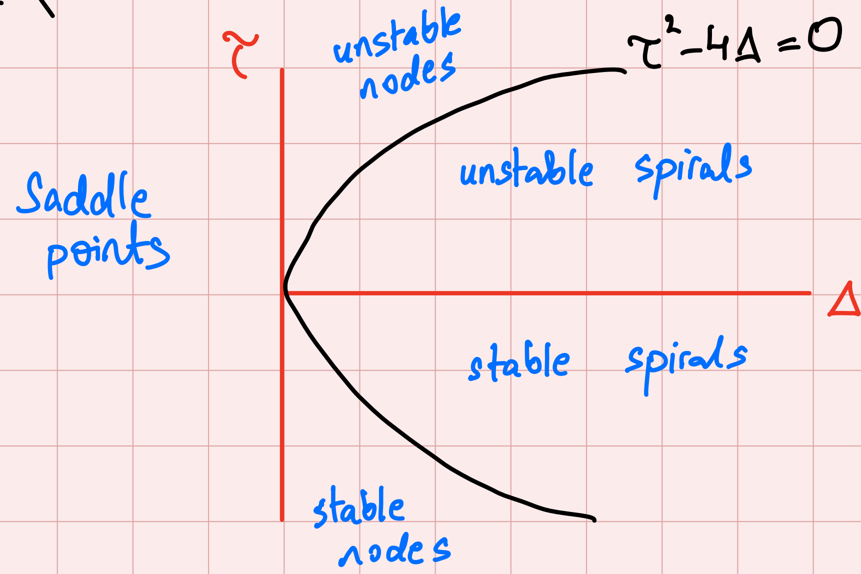
$\lambda_1 = +2$

$\rightarrow \tau = -1$

$\Delta = -6$

A classification of fixed points that occur in any $n=2$ linear system

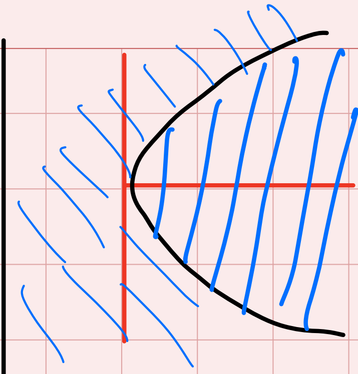
$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$



$0 < Re(\lambda_1) < Re(\lambda_2)$

$Re(\lambda_1) < 0 < Re(\lambda_2)$

$Re(\lambda_1) < Re(\lambda_2) < 0$



$\lambda \in \mathbb{C}$ complex

$\lambda \in \mathbb{R}$ real