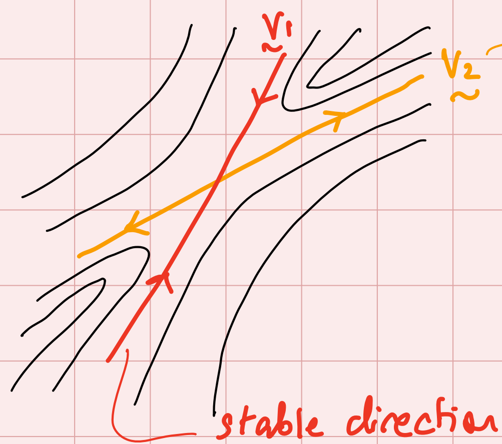
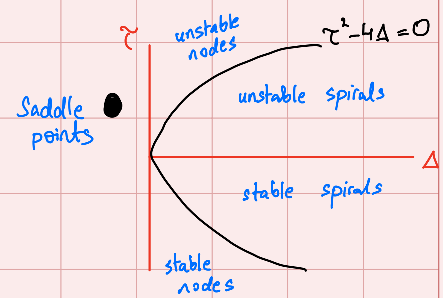
Wed, Feb 19 Lecture 9

Fixed Point Types and eigenvalues / eigenvectors

① Saddle Point

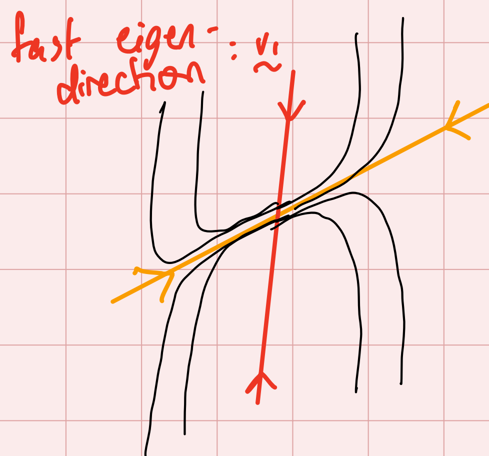
$$\underbrace{\lambda_1}_{\underset{\sim}{v_1}} < 0 < \underbrace{\lambda_2}_{\underset{\sim}{v_2}}$$

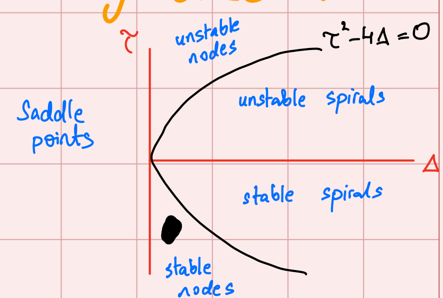




② Nodes

$$\underbrace{\lambda_1}_{\underset{\sim}{v_1}} < \underbrace{\lambda_2}_{\underset{\sim}{v_2}} < 0$$

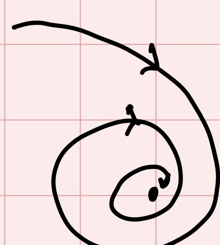


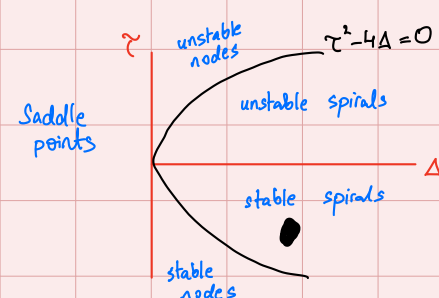


③ Spirals

Re(λ) : exponential decay

Im(λ) : oscillation



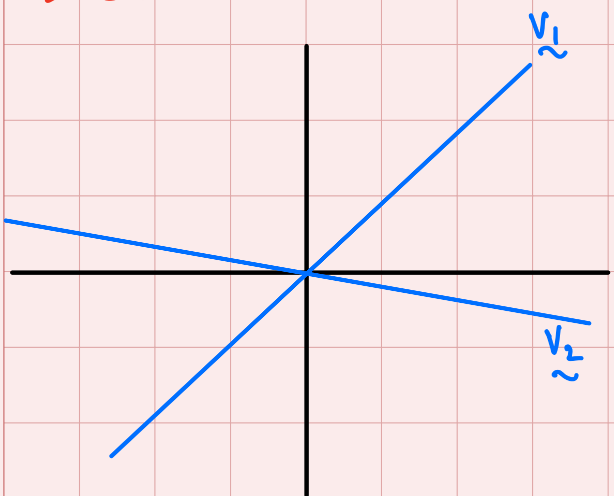


The phase plane (nonlinear $n=2$)

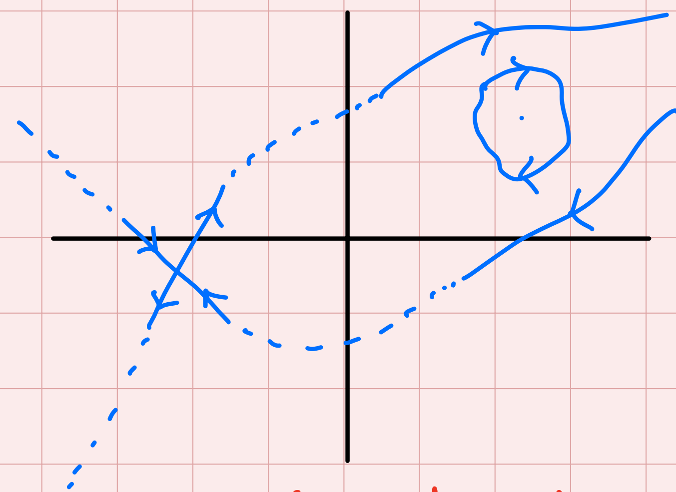
$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$$

$$x_1 = f_1(x_1, x_2)$$
$$x_2 = f_2(x_1, x_2)$$

Linear



Nonlinear



n eigenvectors paint a global picture of the phase plane.

n eigenvectors do not capture global picture.

Features of phase plane:

→ fixed points equilibrium solutions $f(\underset{\sim}{x}^*) = \underset{\sim}{0}$

→ Closed orbits periodic solutions $\underset{\sim}{x}(t+T) = \underset{\sim}{x}(t)$

→ Behavior of $\underbrace{\text{solutions}}_{\text{aka trajectories in this context.}}$ near fixed pts & closed orbits

Linearize !

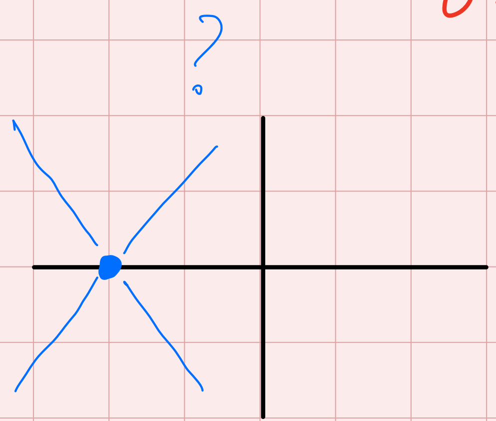
Consider
$$\dot{x} = x + e^{-y}$$
$$\dot{y} = -y$$

$$e^{-y} = 1 - y + \frac{y^2}{2!} - \frac{y^3}{3!} + \cdots$$

$$e^{-y} \approx 1 - y$$

- Find fixed pt : $0 = x + e^{-y}$, $0 = -y$ $\Rightarrow$ $y = 0,\ x = -1$

$$(x^*, y^*) = (-1, 0)$$



$$\dot{x} \approx x + 1 - y$$
$$\dot{y} = -y$$

Define $x + 1 \longrightarrow x'$

NOT Derivative

$$\dot{x}' = x' - y$$
$$\dot{y} = -y$$

$$\begin{bmatrix} \dot{x}' \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\tau = 0,\quad \Delta = -1 \quad \Rightarrow$ saddle

$$\dot{x} = f_1(x, y)$$
$$\dot{y} = f_2(x, y)$$

and $f_1(x^*, y^*) = f_2(x^*, y^*) = 0$

Let $u = x - x^*$

$v = y - y^*$

$$\dot{u} = \dot{x}$$
$$= f_1(x^* + u,\ y^* + v)$$

$$= \underbrace{f_1(x^*, y^*)}_{\text{0 by def. of fixed pt.}} + u \frac{\partial f_1}{\partial x}\bigg|_{\substack{x=x^* \\ y=y^*}} + v \frac{\partial f_1}{\partial y}\bigg|_{\substack{x=x^* \\ y=y^*}} + \underbrace{\text{higher order terms.}}_{\text{ignore}}$$

$$\Rightarrow \boxed{\dot{u} = u \frac{\partial f_1}{\partial x}\bigg|_{\substack{x^* \\ y^*}} + v \frac{\partial f_1}{\partial y}\bigg|_{\substack{x^* \\ y^*}}}$$

by a similar argument,

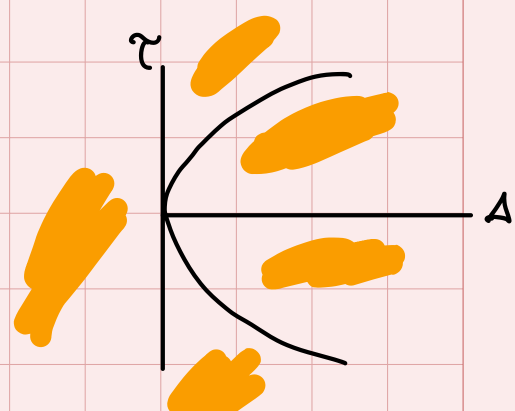
$$\boxed{\dot{v} = u \frac{\partial f_2}{\partial x}\bigg|_{\substack{x^* \\ y^*}} + v \frac{\partial f_2}{\partial y}\bigg|_{\substack{x^* \\ y^*}}}$$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \underbrace{\overbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}}^{A_{ij} = \frac{\partial f_i}{\partial x_j}}\Bigg|_{\substack{x=x^* \\ y=y^*}}}_{A} \cdot \begin{bmatrix} u \\ v \end{bmatrix}$$

Jacobian matrix for the system $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ evaluated at $\underset{\sim}{x}^*$.

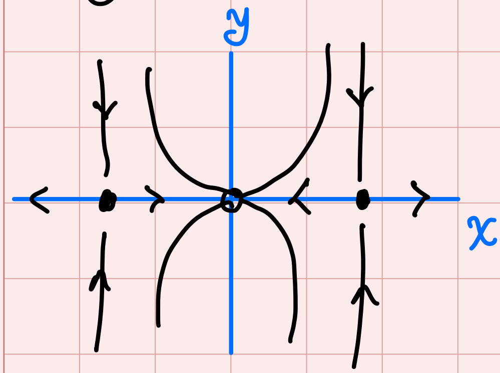
For saddles, spirals & nodes the system $\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix}$ is a good representation of the nonlinear system $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ near (x^*, y^*)

evaluated at (x^*, y^*).



For other types of fixed points, the system $\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix}$ gives a questionable representation of the nonlinear system $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ near (x^*, y^*)

$$\dot{x} = -x + x^3$$
$$\dot{y} = -2y$$



1) Find fixed pts.
2) Characterize each.

What kind of fixed pt. is it?

calculate Jacobian matrix, evaluate at each fixed pt.

where is this matrix on τ-Δ plane?